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AN HISTORICAL INVESTIGATION ABOUT THE DEDEKIND´S CUTS: SOME IMPLICATIONS FOR THE TEACHING OF MATHEMATICS IN BRAZIL

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Abstract: In the training of mathematics teachers in Brazil we can not disregard the historical and epistemological component aiming the transmission of mathematics through a real understanding of the nature of classic concepts and foundamental definitions to Mathematics, whether in the school context or in the academic context. In this sense, the present work addresses a discussion about the introduction and formulation of Dedekind's cut. Such terminology became popular from the work and pioneering research developed by Richard Dedekind (1831 – 1916), although in the set of his contemporaries, as in the case of A. L. Cauchy (1789 – 1857), the proposition of the construction of the real numbers through other notions and others mathematical methods became known. Thus, a historical and epistemological way for the definition of cut is observed and considered. However, Dedekind did not formally answered mainly some of the questions about this notion. The understanding of this epistemological and mathematical process, on the part of the teacher, in which the mathematical intuition and heuristics has an essencial place and requires more attention.

Key words: Historical investigation, Dedekind´s cut, History of Mathematics, Teaching.

1. Introduction

The definition of a solid basis for the foundation of the primitive concepts of Mathematical Analysis did not occur in a progressive, unstoppable or linear way (Alves, 2018a). Indeed, in the context of the History of Mathematics, we recorded a movement of progress and recrudescence of certain assumptions and definitions that have become the object of investigation by several mathematicians over time, because it does not inspire a total recognition and acceptance by researchers and specialists.

On the other hand, when we consider the role of the mathematics teacher (Alves, 2018a; 2018b), an understanding of a mathematical, epistemological, and evolutionary process about mathematical ideas, about intuitive reasoning and some primitive arguments used by mathematicians in the past is essential, often en virtue to partially solve a particular problem. However, is recurrent in Mathematics, inasmuch as some time later the constitution of a solid formal and structural foundation in the sense of replacing preliminarily heuristic and intuitive ideas by solid and structured axiomatic foundation.

Thus, in the present work we present some elements capable of providing an understanding on the evolutionary, historical and epistemological process of formalization of the notion of real number, especially from the thought of Richard Dedekind (1831 – 1916). Richard Dedekind was one of the greatest mathematicians of the nineteenth-century, as well as one of the most important contributors to algebra and number theory of all time (Ferreirós, 2000). Certainly, given the existence of other mathematical forms and analytical methods employed for the formal establishment of a primitive notion for the theory of functions in a real variable, the notion of Dedekind's cut provides an indelible chapter in the History of Mathematics which, despite intuitive and heuristic ideas indicated by Richard Dedekind, the notion of cut, produced either by a rational number or by a non-rational number, finds its roots in ancient Greek thought. Corry (2004) describes some of his academic background.

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> Dedekind studied in his native city of *Braunschweig* and in *Göttingen*. In 1852 he completed his doctoral dissertation, working under the supervision of Gauss. Later in 1854, he habilitated with Bernhard Riemann (1826-1866). During his first years as *Privatdozent* in *Göttingen* he worked in close collaboration with Peter Lejeune Dirichlet (1805-1859). Thus, the decisive influence of the leading mathematicians of the early *Göttingen* tradition marked Dedekind's formative years. This influence was clearly manifest in all of his later work. In 1858 Dedekind was appointed to the ETH in *Zurich*, and in 1862 he returned to *Braunschweig*, were he remained until his death. The notes that Dedekind prepared for his lectures, and which have been preserved in his class, demonstrate that he was a very dedicated and meticulous teacher. Nevertheless, Dedekind neither created a circle of students around himself nor had any single important student. The influence of his ideas came mostly through his interaction and cooperation with leading contemporary mathematicians, such as Heinrich Weber. (Corry, 2004, p. 224).

On the other hand, although we will show in the next sections how certain intuitive ideas and notions have irreversibly affected by the Richard Dedekind´s thinking, we can record his attempt to reduce the role and reliability of heuristic and intuitive ideas in the context of research mathematics, as we can observe in their own explanatory words below.

> In science nothing capable of proof ought to be accepted without proof. Though this demand seems so reasonable yet I cannot regard it as having been met even in the most recent methods of laying the foundations of the simplest science; viz., that part of logic which deals with the theory of numbers. In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought. My answer to the problems propounded in the title of this paper is, then, briefly this: numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things. (Dedekind, 1963, p. 31)

Above in the passage we find the defense with vehemence of the notion of proof and demonstration in view of the consolidation of mathematical notions and reliable results. However, we are still interested in the first objectives and reasons, unspoken reasons that led mathematicians, such as Dedekind, to choose and delineate the notion of cut. Thus, in the subsequent section, we will point out some mathematical properties that were required by Dedekind aiming at the construction of real numbers.

2. Some primitive properties

From the Greek thought, we see the first concerns with the formalization and the mathematical treatment for the forms of measurement. In this sense, we recall the Greek treatment developed for the commensurable quantities which, from a unit denominated as standard to measure, and two quantities can be classified as commensurable. One of the classical Greek problems can be seen from the figure and the geometric construction which produces the corresponding diagonal of a square of unitary side. Here we will not use the symbol $\sqrt{2}$ as an indication that the Greeks did not know its meaning or definition. On the other hand, an investigation in search of the determination of a ration like $\frac{p}{q}$ whose square corresponds to the number $\left(\frac{p}{q}\right)^2 = 2$ becomes more admissible. From the same argument, Niven (1961, p. 43 – 44) affirms that $\sqrt{3}$ and $\sqrt{6}, \sqrt{2}+\sqrt{3}$, $\sqrt[3]{2}$ are irrational numbers. In fact, we suppose that $\sqrt{3}$ were a rational number, say $\sqrt{3} = \frac{a}{b}$, where a and b are integers. Again, as in the $\sqrt{2}$ case, we presume that $\frac{a}{b}$ is in lowest terms, so that not both a and b are divisible by 3. Squaring and simplifying the equation, we obtain $3 = \frac{a^2}{b^2} \leftrightarrow a^2 = 3b^2$. The integer $3b^2$ is divisible by 3; that is, a^2 is divisible by 3. So a itself is divisible by 3, say $a = 3c$, where c is an integer. Replacing a by 3c in the equation $a^2 = (3c)^2 = 9c^2 = 3b^2$, we get: $3c^2 = b^2$. This shows that b^2 is divisible by 3, and hence b is divisible by 3. But we have established that both a and b are divisible by 3, and this is contrary to the presumption that alb is in lowest terms. Therefore $\sqrt{3}$ is irrational. Niven (1961, p. 44) comments that "the proofs of the irrationality of $\sqrt{2}$ and $\sqrt{3}$ depended on divisibility properties ofjntegers by 2 and by 3, respectively, but the corresponding proof for $\sqrt{6}$ can be made to depend on divisibility either by 2 or by 3". For example, if we reason in a similar way to the case of $\sqrt{2}$ proof, we would assume that $\sqrt{6} = \frac{a}{b}$ where the integers a and b are not both even. Squaring, we would obtain $a^{\ 2} = 6b^2$. From a repeated reasoning, Niven (1961) says "Now, $6b^2$ is even, so a^2 is even, so a is even, say $a = 2c$. Then we can write: $a^2 = 6b^2, 4c^2 = 6b^2, 2c^2 = 3b^2$". This tells us that $3b^2$ is even, so b^2 is even, and thus b is even. But a and b were presumed to be not both even, and so $\sqrt{6}$ is irrational. The reader may, as an exercise, deduce the same conclusion by means of a proof which is analogous to the $\sqrt{3}$ proof. For an appreciation of other cases we suggest that the reader consult directly with Niven's (1971) books, in any case, the previous arguments may cover up a certain relation between the numbers that are acquiring the quality and the behavior of non-rational with their corresponding and total correspondence in the line, without mentioning a relation of order that we already know in the rational ones.

From these arguments, here one should note the way of that the Greeks do not acquire a greater understanding of the abstract nature of the number and, in spite of the fact that it does not correspond to a rational number (Gow, 2010), we can not say that they have succeeded precisely in understanding about the existence of a larger set than the rational numbers and which today we call real numbers (Maor, 1987). In figure 1 we observe an elementary construction that proporcinou to determine the existence of an unmeasurable quantity.

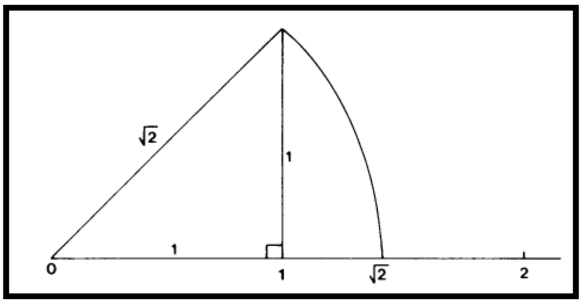

Figure 1. *Maor (1987, p. 45 – 46) comments on the Greek investigation process in order to determine a ratio corresponding to the number* $\sqrt{2}$

Stein (1990) recalls the theory of proportions from the works of Eudoxus and Euclid, which corresponds to a kind of theory of proportions. However, throughout the passage below, we observe that the author indicates some limitations of Greek thought, as regards a solid basis for a primitive arithmetic reasoning.

> The Eudoxean-Euclidean theory of ratio and proportion involves three distinct (interrelated) notions: *number*, *magnitude*, and *ratio*. The notion of magnitude is just presupposed in Euclid's exposition: neither definitions nor explicit assumptions are formulated concerning it, and number, although it is made the subject of a definition, is also in effect simply taken to be understood (for the definition does not provide a basis

> for arithmetical reasoning); but ratio is defined in a remarkably precise and adequate way. The phrase "theory of proportion" is used because the notion of similarity of ratio is crucial (both to the development of the theory, and to the very definition of ratio); and two pairs of magnitudes that have the same ratio are said to be "proportional" (or "in proportion"). (Stein, 1990, p. 166)

In the above passage, we observe some elements produced by the author in spite of a necessity of a consistent arithmetic thought developed by the ancient Greeks (GARDIES, 1984). On the other hand, if we use a current theory, we can easily verify, through the fundamental theorem of arithmetic, that there are no rational solutions to the equation $x^2 = 2$ or $x^2 = 3$. Penney (1972, p. 282) concludes that there are no rational solutions for such an equation, and adds that whatever the solution may not belong to the set of rational numbers. In this example, we can observe the signs of deficiencies present in the set of rational numbers when we consider their disposition in a straight line. In this case, Richard Dedekind expressed great interest in finding and studied such this deficiency as we can observe his statements in the following passage:

> Of the greatest importance, however, is the fact that in the straight line *L* there are infinitely many points which correspond to no rational number. If the point *p* corresponds to the rational number *a*, then, as is well known, the length *0 p* is commensurable with the invariable unit of measure used in the construction, i. e., there exists a third length, a socalled common measure, of which these two lengths are integral multiples. But the ancient Greeks already knew and had demonstrated that there are lengths incommensurable with a given unit of length, e. g., the diagonal of the square whose side is the unit of length If we layoff such a length from the point () upon the line we obtain an end-point which corresponds to no rational number. Since further it can be easily shown that there are infinitely many lengths which are incommensurable with the unit of length, we may affirm: The straight line *L* is infinitely richer in point-individuals than the domain *R* of rational numbers in number individuals. (Dedekind, 1963, p. 9)

Richard Dedekind was one of the mathematicians responsible for progress and fundamentalism for arithmetic, from the proposition of the emblematic notion of cut. Spivak (1967, p. 688) comments that the real numbers constructed in this chapter (of his book) might be called "the algebraist's real numbers," since they were purposely defined so as to guarantee the least upper bound property, which involves the ordering <, an algebraic notion. The real number system constructed in the next problem might be called "the analyst's real numbers," since they are devised so that Cauchy sequences will always converge. Dugac (2003, p. 160) comments on Dedekind's early experiences as an autonomous teacher, In the Polytechnic School, during the teaching of a course of differential calculus. He points to Dedekind's feeling of dissatisfaction with the use of certain arguments whose geometric intuition has become irreplaceable.

Ferreiros (2000, p. 82) recalls that R. Dedekind obtained his habilitation in 1854 and, until the presentation of his thesis did not reveal outstanding mathematical talents. Only in the period of formation in 1855 and 1858 did Dedekind stand out, with a thought influenced by the mathematician Johann Peter Gustav Dirichlet. (1805-1859) and Georg Friedrich Bernhard Riemann (1826 – 1866) (Sinaceur, 1990). In the figure below, we can see the young mathematicien.

Spivak's considerations should help us in the demarcation of the field of our discussion, given the existence of various methods of constructing real numbers (Hafnner, 2014; Nagumo, 1976; Niven 1956). Thus, throughout the work, in order to delineate a perspective capable of promoting a perspective of historical investigation and that can influence the actions of the teacher of Mathematics in Brazil, we will restrict ourselves to the method proposed by Dedekind, with the intent to understand the intuitive bases that have influenced the constitution of the notion of cut.

Figure 2. *Dedekind as a teacher at the Polytechnic School in Germany (Ferreirós, 2000)*

3. The cut´s notion

Courant & Robbins (1996, p. 72) comment that "philosophically, Dedekind´s definition of irrational numbers involves a rather high degree of abstraction, since it´s places no restrictions on the nature of the mathematical law which defines the two classes A and B. A more concrete method of defining a real *continuum* number is due to George Cantor (1845 – 1918)". Immediately afterwards we present an important testimony of Dedekind himself and his preoccupations with teaching, the necessity of the basis of Mathematical Analysis and the establishment of solid foundations for Arithmetic.

> As professor in the Polytechnic School in *Zurich* I found myself for the first time obliged to lecture upon the elements of the Differential Calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis. (Dedekind, 1963, p. 1 – 2).

Further on, we note the long periods of reflection needed for Richard Dedekind to finally present some of primitive ideas that contributed to the description of the notion of cut. It is interesting to note his hesitations as a renowned mathematician in the sense of disseminating his results with other teachers and his students.We can observe below his words.

> I succeeded November 24, 1858, and a few days afterward I communicated the results of my meditations to my dear friend Durege with whom I had a long and lively discussion. Later I explained these views of a scientific basis of arithmetic to a few of my pupils, and here in *Braunschweig* read a paper upon the subject before the scientific club of professors, but I could not make up my mind to its publication, because in the first place, the presentation did not seem altogether simple, and further, the theory itself had little promise. (Dedekind, 1963, p. 2).

Dedekind sought to draw inspiration from certain properties of the line, when we intend to put in correspondence and dispose the set of real numbers on it. The property of continuity of the straight line presents itself as a geometric, perceptual and qualitative character that Dedekind sought to develop a formal treatment. Below we check his expectations.

> If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary

> that the instrument *R* constructed by the creation of the rational num bers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same continuity, as the straight line. (Dedekind, 1963, p. 2).

Ferreirós (2000, p. 26) confirms an expedient employed by Dedekind who proved to be greatly influenced by the conversations with his students in mathematics teaching occasions. “The notion of set was absent from Dedekind´s work up to 1855. Even when we dealt with foundations issues. However, it was conspicuously both in his algebraic work of 1856 – 58 ans his theory of irrational numbers, which dates from 1858”. (Ferreirós, 2000, p. 77). On the other hand, the theory of irrational numbers required an adequate understanding of various geometric properties, and in them, we found that intuition, regarded as an unwanted element for a formal theory, but intuition was always presented as an aid to Dedekind's thought as we can observe in one of his following questions.

> The above com parison of the domain *R* of rational numbers with a straight line has led to the recognition of the existence of gaps, of a certain incompleteness or discontinuity of the former, while we ascribe to the straight line completeness, absence of gaps, or continuity. In what then does this continuity consist? (Dedekind, 1963, p. 10).

This question required considerable mental effort and reflection for R. Dedekind. Indeed, “for a long time I pondered over this in vain, but finally I found what I was seeking. This discovery will, perhaps, be differently estimated by diferente people; the majority may find its substance very commonplace” (Dedekind, 1963, p. 11).

In Dedekind's own words, we observe some necessary period of reflection and deep investigation with the intention of establishing a set of elements capable of characterizing the nature of a new mathematical conceptual entity that was intuitively perceived through certain topological properties such as the notion of continuity from the oriented axis. Finally, we note an important principle pointed out by Dedekind.

> It consists of the following. In the preceding section attention was called to the fact that every point *p* of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i. e., in the following principle: If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. (Dedekind, 1963, p. 10).

On the other hand, we show the heuristic and intuitive character assumed by Dedekind, when he tried to describe the process of seperation of a straight line into two still continuous portions. In addition, we observe a notion of order that, according to Dedekind, makes it possible to compare two elements of these two special classes.

> As already said I think I shall not err in assuming that everyone will at once grant the truth of this statement; the majority of my readers will be very much disappointed in learning that by this commonplace remark the secret of continuity is to be revealed. To this I may say that I am glad if everyone finds the above principle so obvious and so in harmony with his own ideas of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power. The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line. (Dedekind, 1963, p. 10 - 11).

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> In Section I it was pointed out that every rational number *a* effects a separation of the system *R* into two classes such that every number a_1 of the first class A_1 is less than every number a_2 of the second class A_2 *;* the number *a* is either the greatest number of the class A_1 or the least number of the class A_2. If now any separation of the system *R* into two classes A_1, A_2, is given which possesses only *thi's* characteristic property that every number al in A_1 is less than every number a_2 in A_2, then for brevity we shall call such a separation a *cut* [*Schnitt*] and designate it by (A_1, A_2 *)*. (Dedekind, 1963, p. 11)

Above we see Dedekind's description of the emblematic notion of cut. It should be noted that the term "*cut*" refers basically to the process of separation of the line into two classes and now is indicated by (A_1, A_2 *)*, while the term "*section"* refers to the process of separation of the set of rational numbers. One of Dedekind's conclusions was to note the fact that not all sections are produced by rational numbers. Thus, Dedekind showed that there are infinite *sections* produced by non-rational numbers. From this fact, Dedekind assumed that whenever we are before a *section* originated and produced by a number that is not rational, the creation occurs the emergence of a new number, whose natural difference is distinguished from a rational number.

Henle (2012, p. 24 – 25) comments that "there is no rational number in Q whose square is 2. This is what we mean by a hole: the rationals are missing the square root of 2, a number we can approximate to as many places as we like, that occupies, it seems, a definite place along a number line, but cannot be expressed as a fraction $\frac{p}{q}$ where p and q are integers". We see in this commentary the idea of approximation that differs from the argument used by Dedekind. It should be noted that some expressions used by Dedekind were considered by some authors to be imprecise and vague in the mathematical sense, although, from Dedekind's own words, it is observed that demonstration and formal proof function irreplaceably in Mathematics.

In the figure 3[1] we visualize a geometric interpretation proposed by Henle (2012) for a particular case.

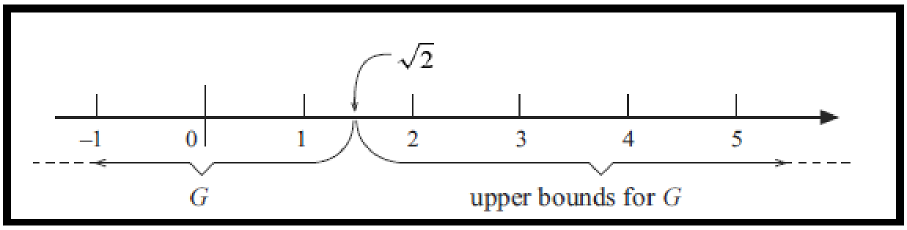

Figure 3. *Henle (2012, p. 25) describes a geometric interpretation for the notion of Dedekind´s cut.*

Let us now see his fundamental argument, which consists in demonstrating that there are infinite *cuts* not produced by rational ones (Dedekind, 1963, p. 13). To this end, Dedekind considered D a positive integer, different from of any perfect square, then there is a positive integer $\lambda > 0$, such that: $\lambda^2 < D < (\lambda+1)^2$. If we consider the second class A_2, consisting of all positive rational numbers a_2, whose square is larger than D and the first class A_1, constituted by all other rational numbers

[1] "Carl Friedrich Gauss (1777–1855) used the idea of least upper bounds informally, which was a step forward in the development of the real numbers, but he did not provide a construction of the real numbers; he had the older notion of the real numbers as varying continuously, and took that as the basic intuitive idea of real analysis". (BLOCH, 2010, p. 57)

a_1, this separation forms the section (A_1, A_2). In this way, every number a_1 is less than any number a_2. Notice that if it occurs that $a_1 \leq 0$ then it is obvious that less than every element a_2, Therefore, we define that $0 < d \leq a_2$.

Dedekind, from the elements indicated above, sought to show that the section (A_1, A_2) is not produced by any rational number, that is, there is no rational whose square is equal to D. Consequently, it has proved that there are an infinite number of sections that are not originated by rational numbers. To do so, let's take the previous inequality $\lambda^2 < D < (\lambda+1)^2$, where D is a positive integer, different from a perfect square, and $\lambda > 0$ positive integer.

Dedekind found that there is no rational number whose square is equal to D. In fact, let's assume, provisionally, that there is a rational number $D = \left(\frac{t}{u}\right)^2$ and the positive integers that we will indicate by $t, u > 0$, so that, we have: $\left(\frac{t}{u}\right)^2 = \frac{t^2}{u^2} = D \leftrightarrow t^2 = D \cdot u^2 \leftrightarrow t^2 - D \cdot u^2 = 0$. From this equation, we take $t^2 - D \cdot u^2 = 0$ admitted that 'u' it is the smallest positive integer having the property that its square multiplied by D, and that can be converted (produces) the square of a given integer indicated by t (*). But since we know that $\lambda^2 < D < (\lambda+1)^2$ and, taking the necessary substitution, we have $\lambda^2 < \frac{t^2}{u^2} < (\lambda+1)^2 \therefore \lambda^2 u^2 < t^2 < (\lambda+1)^2 u^2$. Therefore, Dedekind found that $\lambda \cdot u < t < (\lambda+1) \cdot u$. That is, we have to: $\lambda \cdot u < t < (\lambda+1) \cdot u = \lambda u + u$.

Let us note, however, that we have $t < u + \lambda u$ and so it comes that $u' := (t - \lambda \cdot u) < u$ and $\lambda \cdot u < t \leftrightarrow 0 < t - \lambda \cdot u = u'$ and we find an integer $u' := t - \lambda \cdot u$ strictly smaller than u. In the next step, Dedekind considered the number $t' := Du - \lambda t = \left(\frac{t^2}{u^2}\right) \cdot u - \lambda \cdot t = \frac{t^2}{u} - \lambda \cdot t = \frac{t^2 - \lambda \cdot t \cdot u}{u}$. Let us note, however, that we must have the equality $t^2 - \lambda \cdot t \cdot u = t \cdot (t - \lambda \cdot u)$ and since $\lambda \cdot u < t < (\lambda+1) \cdot u$, we can stil write $t^2 - \lambda \cdot t \cdot u = t \cdot (t - \lambda \cdot u) > 0$. Therefore, the positive integer $t' := Du - \lambda t > 0$ and provides to verify the following property indicated below and indicating some algebraic manipulations:

$$(t')^2 - D \cdot (u')^2 = \left(Du - \lambda t\right)^2 - D \cdot \left(t - \lambda u\right)^2 = D^2 u^2 - 2Du\lambda t + \lambda^2 t^2 - D\left(t^2 - 2\lambda\, t u + \lambda^2 u^2\right) =$$
$$= D^2 u^2 - 2Du\lambda t + \lambda^2 t^2 - Dt^2 + 2\lambda\, t uD - \lambda^2 u^2 D = -D(t^2 - D \cdot u^2) - 2Du\lambda t + \lambda^2 (t^2 - D \cdot u^2) + 2\lambda\, t uD$$
$$= -D \cdot 0 - 2Du\lambda t + \lambda^2 \cdot 0 + 2\lambda\, t uD = 0 - 2Du\lambda t + 2\lambda\, t uD = 0, \text{ this is, } (t')^2 - D \cdot (u')^2 = 0$$

Such equality $(t')^2 - D \cdot (u')^2 = 0$, since that $u' := (t - \lambda \cdot u) < u$ immediately opposes the choice of the minimum integer in (*) which corresponds to another equation $t^2 - D \cdot u^2 = 0$, since Dedekind determined a pair of integers $u' := (t - \lambda \cdot u) < u$ and $t' := Du - \lambda t > 0$, such that $D = \left(\frac{t'}{u'}\right)^2$. Consequently, there will be no rational under these conditions. From the previous property, Dedekind found that the square of any rational number x^2 or is less than D or will be greater than D, still, it can never be the same as D. Returning to the set or ordered pair indicated (A_1, A_2), Dedekind demonstrated that in class A_1 there is no maximum element, whereas in the class A_2 there is no minimum element. As we will see later, such property is imperative for the formal conditions of a cut to be fulfilled.

For this, given any rational $x \in Q$, Dedekind (1963, p. 14) considered that $y = \frac{x(x^2+3D)}{3x^2+D}$. In this way, we write $3x^2y + Dy = x^3 + 3xD \therefore (3x^2y + Dy - x^3 - xD) = 2xD$ and then, by adding, on both sides, the term indicated by ($-2x^3$), it also follows that:

$$(3x^2y + Dy - 3x^3 - Dx) = (2xD - 2x^3) \leftrightarrow (y-x)(3x^2+D) = 2x(D-x^2) \leftrightarrow y - x = \frac{2x(D-x^2)}{3x^2+D}.$$

It also considered the following differences $y - x = \frac{2x(D-x^2)}{3x^2+D}$ and $y^2 - D = \frac{(x^2-D)^3}{(3x^2+D)^2}$. Let's see that the last equality indicated by $y^2 - D = \frac{(x^2-D)^3}{(3x^2+D)^2}$ can be verified as follows: $y^2 - D = \left(\frac{x(x^2+3D)}{3x^2+D}\right)^2 - D = \frac{x^2(x^2+3D)^2}{(3x^2+D)^2} - D = \frac{x^2(x^2+3D)^2 - D(3x^2+D)^2}{(3x^2+D)^2}$. Its follows that:

$$y^2 - D = \frac{x^2(x^2+3D)^2 - D(3x^2+D)^2}{(3x^2+D)^2} = \frac{x^6 + 6x^4D + 9D^2x^2 - D(9x^4 + 6x^2D + D^2)}{(3x^2+D)^2} =$$

$$\frac{x^6 + 6x^4D + 9x^2D^2 - 9x^4D - 6x^2D^2 - D^3}{(3x^2+D)^2} = \frac{x^6 + 3x^2D^2 - 3x^4D - D^3}{(3x^2+D)^2} = \frac{(x^2-D)(x^4 - 2x^2D + D^2)}{(3x^2+D)^2} = \frac{(x^2-D)^3}{(3x^2+D)^2}.$$

Thus, for the following numbers indicated in the form $y = \frac{x(x^2+3D)}{3x^2+D}$, $y - x = \frac{2x(D-x^2)}{3x^2+D}$ and $y^2 - D = \frac{(x^2-D)^3}{(3x^2+D)^2}$ Dedekind showed that there is no maximum in class or set described by $A_1 = \{y \in Q \mid y^2 < D\}$ (there is no maximum element), and by a similar reasoning, whereas in the class $A_2 = \{y \in Q \mid y^2 > D\}$ there is no minimum element.

In fact, we have already seen that (a) $x^2 < D$ ou (b) $x^2 > D$ and never the exactly equation $x^2 = D$. In the first case, if we had to $x \in A_1$, then $x^2 < D$. However, for this element $x \in A_1$ we take or verify that we have the following inequation $y = \frac{x(x^2+3D)}{3x^2+D} > x$, because if it happened that $y = \frac{x(x^2+3D)}{3x^2+D} < x \leftrightarrow x(x^2+3D) = \mathrm{x}^3 + 3xD < 3x^3 + xD \leftrightarrow 2xD < 2x^3$, that is, it would occur that $D < x^2$ which is a contradiction in the case of $x^2 < D$ (a). Therefore, we can always consider the following element element $y = \frac{x(x^2+3D)}{3x^2+D} > x$ and in away that we have $y^2 - D = \frac{(x^2-D)^3}{(3x^2+D)^2} < 0 \therefore y^2 - D < 0$, that is, it is worth $y^2 < D$, $y \in A_1$, with $x > 0$ (c.q.d).

Therefore, the class $A_1 = \{y \in Q \mid y^2 < D\}$ does not admit of a maximum element. On the other hand, if we suppose that (b) $x^2 > D$ and $x \in A_2$, we can verify that an element can be found $y < x, y \in A_2$, that is, the class A_2 does not admit of a minimum element. In fact, we must always $y = \frac{x(x^2+3D)}{3x^2+D} < x$, for $x^2 > D$.

Otherwise, we have $y = \frac{x(x^2+3D)}{3x^2+D} > x \leftrightarrow x^3 + 3xD > 3x^3 + xD \leftrightarrow 2xD > 2x^3 \leftrightarrow D > x^2$, however, we are exactly in the case of $x^2 > D$. Thus, given the element $x^2 > D$ and $x \in A_2$, considering the fact that $y = \frac{x(x^2+3D)}{3x^2+D} < x$ and $y^2 - D = \frac{(x^2-D)^3}{(3x^2+D)^2} > 0 \leftrightarrow y^2 > D$, that is, $y \in A_2$, provided that $y = \frac{x(x^2+3D)}{3x^2+D} < x$. No minimum element is allowed.

As we have mentioned, the preceding argument shows that not all sections are produced by rational numbers. In this case, Dedekind mentioned that such a section corresponds to a new number called irrational. In fact, Dedekind declares that "whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an irrational number "a", which we regard as completely defined by this cut (A_1, A_2) we shall say that the number "a" corresponds to this cut, or that it produces this cut". At this point we observe an important relation intended by Dedekind and that corresponds to say or observe that if a certain cut corresponds to a rational element when we can verify the existence of a maximal element in A_1 and the existence of a minimal element in A_2. If, as Lefebvre (1998, 24) points out, we are faced with an irrational element determined by the cut (A_1, A_2).

Moreover, we note that for every rational $x \in Q$ we can define the particular cut $A_x = \{q \in Q \mid q < x\}, \mathrm{B}_x = \{q \in Q \mid q > x\}$. In this case, we observe $x \in Q - A_x \bigcup B_x$. On other other hand, if we consider another cut (A, B) de modo que $Q - A \bigcup B \neq \varnothing$, then we must have that $A_x = A, B_x = B$ and it´s enough to use de cut´s definition for proving it.

Dedekind was also responsible for introducing some notions of infinite sets. The mathematician Richard Dedekind suggested an alternative, but equivalent, definition of what it means for a set to be infinite; the term used is that the set is Dedekind infinite, and this means that the set can be put into one-to-one correspondence with one of its proper subsets. Theorem below shows that this is the same property as being infinite. Dedekind is generally acknowledged to be the father of classical set-theoretic algebra and, by every measure logical and mathematical, was no constructivist. (McCARTY, 1996, p. 54).

Theorem: A set S is infinity if and only if S is Dedekind-infinity.

Proof. We suggest to the reader consulte Conway (1976, p. 234).

Moreover, as we can see from Conway's (1976) considerations, despite Dedekind's apparent pretension to assume totally formal and axiomatically valid assumptions, Dedekind received a number of criticisms[2], although many of his arguments were rescued *a posteriori* and verified or generalized.

> His method produces a logically sound collection of real numbers (if we ignore some objections on the grounds of ineffectivity, etc.), but has been criticised on several counts. Perhaps the most important is that the rationals are supposed to have been already constructed in some other way, and yet are "reconstructed" as certain real numbers. The distinction between the "old" and "new" rationals seems artificial but essential. (Conway, 1976, p. 3 – 4).

Dedekind's definition expresses our geometrical intuition of the *continuum*, which has been so deeply rooted since the days of classical antiquity. This intuition tells us that the points of a straight line are defined by "the bisection of a line into two parts". (Ebbinghaus et all, 1991, p. 34). Among some notions that received a little formal treatment by Dedekind we point out the notion of density that, for Leibniz, for example, the continuity of points on a line is linked to the notion of density. The density of two points means, intuitively that if we take any two points it is always possible to determine a third point, of the same nature between these two initials. For example, if we take the rational points 0 and 1, we can always find the set of numbers: $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\ldots,\frac{n}{n+1},\cdots$ between 0 and 1. If we consider, the numbers 0 and $\frac{1}{2}$ we can observe: $\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{4}{9},\frac{5}{11},\ldots,\frac{n}{2n+1},\cdots$. And, a last particular case, if we take the numbers 0 and $\frac{1}{4}$ we can get the infinit set indicated: $\frac{1}{5},\frac{2}{9},\frac{3}{13},\frac{4}{17},\frac{5}{21},\ldots,\frac{n}{4n+1},\cdots$. However, the curious fact is that the density of the set of rational numbers does not guarantee the continuity and some propertie of the line. In the theorem below we will formalize the previous arguments from the Penney (1972).

Theorem: Let r and s rational numbers with $r < s$. Then there are infinitely many rational numbers between rand s. (Penney, 1972, p. 284).

Proof. Choose integers m and n such that $0 < m < n$. Then $0 < \frac{m}{n} < 1$. On the other hand, since we have $r < s$, and $r, s \in Q$ and $s - r \in Q_+^*$. So we have $0 < \frac{m}{n}\cdot(s-r) < 1\cdot(s-r)$. And hence $r < \frac{m}{n}\cdot(s-r) + \mathrm{r} < s$. Since $\frac{m}{n}$ is a rational number, so is their product, and so is their sum indicated by $\frac{m}{n}\cdot(s-r)+\mathrm{r} \in \mathrm{Q}^*$. But there are infinitely many choices of integers m and n such that $0 < m < n$, and m and n can be chosen so as to give infinitely many different values of m/n, and thus infinitely many different values of $\frac{m}{n}\cdot(s-r)+\mathrm{r}$ and which lies between r and s. This establishes the theorem□

In order to conclude this section, we recover important arguments discussed by Mccarty (1996), where he confers three mysteries related to Dedekind's point of view and his activity as a professional

[2] "Dedekind also gave a proof of the existence of an infinite set, but it was based on the inconsistent concept of the set of all sets". (Ebbinghaus, 1991, p. 16)

mathematician. Although extensive it is revealing and explains, at least in part, the reason for his choices and his point of view on mathematics itself.

> […] and I have three. Pending their resolutions, any mathematical portrait of Richard Dedekind remains unfinished. For the mysteries are mathematical veils across the face of Dedekind's work. The first of the three involves a "proof" of something most set theorists do not prove and a conclusion most people would not allow. What set theorists normally do not prove is the axiom of infinity, the assertion that there exists an infinite set. What Dedekind offers us seems to be a mysterious "proof" of exactly that. […] The second mystery revolves around Dedekind's famous essay on continuity, *Stetigkeit und irrationale Zahlen*. We uncover the mystery by asking, "How can Dedekind claim to have captured - in his definition of 'real number' - the essence of the continuum and, at the very same time, describe for us a continuum which we cannot see to satisfy that definition?" We suppose that, in the "Continuity" essay, Dedekind first unveils his definition of the *continuum*, the one formulated in terms of Dedekind cuts. We do not often remark upon the fact that, before setting out that definition, Dedekind attempts to motivate it by assuming the existence and examining the character of another, seemingly distinct continuum. […] On the face of it, the third mystery seems to be one of historical classification rather than mathematical individuation. Dedekind is generally acknowledged to be the father of classical set-theoretic algebra and, by every measure logical and mathematical, was no constructivist. Yet, Dedekind avers - often and throughout his writings - that mathematical entities are not mind-independent, Platonistic abstracta, but are literally geistig or mental. More troubling still, Dedekind seems to think them not sempiternal but brought into being by discrete mental episodes, acts he calls *freie Schopfung* - free creation. (Macarty, 1995, p. 53 – 54)

In the next section we will discuss some related elements like methods and approaches proposed for the construction of the real numbers, with emphasis to the method proposed by Dedekind. It is good to remember that Dedekind's ideas were strongly influenced by Riemann's conceptions of the continuum, when the latter thought in terms of the foundations for Riemannian geometry. "Dedekind's style for the creation of irrational numbers was inspired by Riemannian innovation [...]" (Sinaceur, 1990, p. 230 – 231).

4. The constructions of real numbers

"It was Dedekind, Weierstrass, and others in the 19th century who eventually restored Greek standards of rigor" (Stillwell, 2010, p. 55). On the other hand, we can not disregard the contributions of Greek thought, which definitively faced a problem not only mathematical but, above all, philosophical, when dealing with immeasurable magnitudes. Below, Stillwell (2010) confirms the work of some generations of mathematicians, centuries later, in order to formalize intuitive and heuristic ideals of the ancient Greeks.

> However, it is unlikely that the Pythagoreans would have viewed $\sqrt{2}$ has a "limit" or seen the sequence as a meaningful object at all. The most we can say is that, by stating a recurrence, the Pythagoreans *implied* a sequence with limit $\sqrt{2}$, but only a much later generation of mathematicians could accept the infinite sequence as such and appreciate its importance in defining the limit. (Stillwell, 2010, p. 55).

We then observe that Stillwell (2010) restores the role of intuition as a human ontological faculty explored both in the context of ancient Greek thought as well as in the context of mathematical research in the twentieth century, where the foundation for arithmetic were established. We observe that the mathematical intuition, although recurrently required, was denied by several mathematicians and among them, Richard Dedekind himself.

> The theory of proportions was so successful that it delayed the development of a theory of real numbers for 2000 years. This was ironic, because the theory of proportions can be used to define irrational numbers just as well as lengths. It was understandable though, because the common irrational lengths, such as the diagonal of the unit square, arise from constructions that are intuitively clear and finite from the geometric point of view. Any *arithmetic* approach to $\sqrt{2}$, whether by sequences, decimals, or continued fractions, is infinite and therefore less intuitive. Until the 19th century this seemed a good reason for considering geometry to be a better foundation for mathematics than arithmetic. Then the problems of geometry came to a head, and mathematicians began to fear geometric intuition as much as they had previously feared infinity. There was a purge of geometric reasoning from the textbooks and industrious reconstruction of mathematics on the basis of numbers and sets of numbers. (Stillwell, 2010, p. 56).

Henle (2012) points out the philosophical vies and that actually surpasses the limits of mathematics itself when we plunge deeply into the ontological understanding of real numbers. The considerations of Henle (2012) confirm in fact that we are not dealing with discussing a subject of easy approach or easy epistemological description.

> That all versions of the reals are algebraically and geometrically isomorphic does not necessarily answer the question: what is a real number? Some readers will be dissatisfied that different constructions result in such different kinds of entities. To them a number like $\sqrt{2}$, for example, should be a definite thing, and not an equivalence class of Cauchy sequences or a Dedekind cut. For some the nature of the real numbers is not settled by these constructions; it remains a problem in the Philosophy of Mathematics. (Henle, 2012, p. 47).

Harrison (1998) describes the contribution of several mathematicians who point out and indicate various approaches and methods for constructing real numbers. The importance of this fact commented on by Harrison (1998) confirms the indispensability of a solid and reliable foundation for a fundamental notion for the whole theory of functions in a real variable. On the other hand, we recall some methods considered by Harrison (1998) as the method of the positive quantities and, from its characterization Nagumo (1976) derived the positive system of real numbers. Then, the extension of the system of positive real numbers to the whole system of real numbers can be easily carried out (NAGUMO, 1976, p. 1). We will illustrate, a little further on, we will discuss some of these properties in the sense of comparing with Dedekind's approach.

> All the methods are conceptually simple but the technical details are substantial, and most general textbooks on analysis, e.g. Rudin (1976), merely sketch the proofs. A pioneering monograph by Landau (1930) was entirely devoted to the details of the construction (using Dedekind cuts), and plenty of similar books have followed, e.g. those by Thurston (1956) (Cauchy sequences), Roberts (1962) (Cauchy sequences), Cohen and Ehrlich (1963) (Cauchy sequences), Lightstone (1965) (positional expansions), Parker (1966) (Dedekind cuts) and Burrill (1967) (positional expansions). Other discussions that survey more than one of these alternatives are Feferman (1964), Artmann (1988) and Ebbinghaus et al. (1990). A very recent collection of papers about the real numbers is Ehrlich (1994). (Harrison, 1998, p. 16).

Harrison (1998) describes the contribution of several mathematicians who point out and indicate various approaches and mathematical methods for constructing real numbers. In the figure 4 below, the Harrison comments on construction methods involving: positional expansions, Dedekind cuts, Cauchy sequences. Moreover, Harrison (1998, p. 16) says that “before we focus on the choice, we should remark that there are plenty of other methods, e.g. continued fractions, or a technique due to Bolzano based on decreasing nests of intervals”. And, a little later, he adds that:

> A more radical alternative (though it is in some sense a simple generalization of Dedekind's method), giving a bizarre menagerie of numbers going way beyond the reals, is given by Conway (1976). As it stands, the construction is hard to formalize, especially in type theory, but Holmes (1998) has formalized a variant sufficing for the reals. Furthermore, there are

> some interesting methods based on the 'point free topology' construction given by Johnstone (1982). A detailed development using the idea of an intuitionistic formal space (Sambin 1987) is given by Negri and Soravia (1995). This technique is especially interesting to constructivists, since many theorems admit intuitionistic proofs in such a framework, even if their classically equivalent point-set versions are highly nonconstructive. For example, there is a constructive proof by Coquand (1992) of Tychonoff's theorem, which is classically equivalent to the Axiom of Choice. (Harrison, 1998, p. 16)

In the righ side, Ehrlich (1994, p. viii) explains the Conway´s method that "further expounds on these difficulties and he also provides an overview of his novel theory of real numbers".

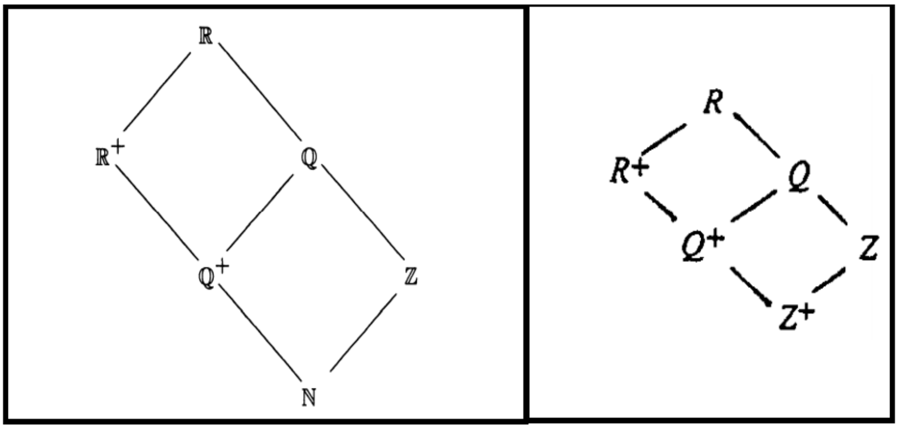

Figure 4. *Harrison (1998) comments a set of mathematical methods for constructing the real numbers and, in the right side, Ehrlich (1994) explains some modified diagram*

Harrison (1998) describes the method proposed by Dedekind involving several axiomatic hypotheses that are not completely satisfactory that give rise to the notion of cut. We observe his explanation and critics below.

> A method due to Dedekind (1872) identifies a real number with the set of all rational numbers less than it. Once again this is not immediately satisfactory as a definition, but it is possible to give a definition not involving the bounding real number that, given the real number axioms, is equivalent. We shall call such a set a cut. (Harrison, 1998, p. 20)

Moreover, Harrison (1998) explains that the method proposed by Cantor involves overly abstract ideas and, we add, can find the need to use a heuristic and local reasoning or argument, as we can see in Dedekind's construction of cuts.

> Cantor's method admits of abstraction to more general structures. Given any metric space, that is, a set equipped with a 'distance function' on pairs of points (see later for formal definition), the process can be carried through in essentially the same way. This gives an isometric (distance-preserving) embedding into a complete metric space, i.e. one where every Cauchy sequence has a limit. Harrison (1998, p. 23)

On the other hand, we can not ignore some criticism and mistrust manifested by some mathematicians, according to the argument that some elements presented in his theory have not been sufficiently explained and formalized, namely some operations involving the cuts, such as: $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$, $\sqrt{2}\cdot\sqrt{2}=2$ that (in Dedekind's opinion) had never been rigorously proved before. Such proofs are possible, but still not trivial (Stillwell, 2010, p. 58). Harrison (1998) explains below some problems related to the Dedekind´s method and lack of mathematical consistency.

> The two cuts X and Y extend to $-\infty$, so there will exist products of these large and negative numbers that are arbitrarily large and positive. Therefore the set is not a cut. This difficulty is usually noted in sketch proofs given in books, but to carry through in detail

> the complicated case splits they gloss over would be extremely tedious. (Harrison, 1998, p. 19)

Harrison (1998) comments on the method of construction of the real by means of positional expansions, whose intuitive vies, as the author himself indicates, are more evident, although it presents certain aspects that require greater formalization. Cauchy's method stands out because of the power of abstraction and the need not to construct any *a priori* set, as in the case of rational ones. We hold below important distinctions for the both mathematical methods.

> It seems that using positional expansions is a promising and unfairly neglected method. As stressed by Abian (1981) and others, the idea of positional expansions is very familiar, so it can be claimed to be the most intuitive approach. However the formal details of performing arithmetic on these strings is messy; even the case of finite strings, though not really very difficult, is tiresome to formalize. Cauchy's method is quite elegant, but it does require us to construct the rationals first, and what's more, prove quite a lot of 'analytical' results about them to support the proofs about Cauchy sequences. It is also necessary to verify that all the operations respect the equivalence relation. Thus, when expanded out to full details, it involves quite a lot of work. The Dedekind method involves a bit of work verifying the cut properties, and again we have to construct the rationals first. (Harrison, 1998, p. 21).

On the other hand, Conway (1976) adds some metric properties and the notion of metric completeness of space, in that an expansion of the set where the metric has been defined. However, there are already many problems in the sense of making the final expansion coherent, as we can see below.

> The reader should be cautioned about difficulties in regarding the construction of the reals as a particular case of the completion of a metric space. If we take this line, we plainly must not start by defining a metric space as one with a real-valued metric! So initially we must allow only rational values for the metric. But then we are faced with the problem that the metric on the completion must be allowed to have arbitrary real values! Of course, the problem here is not actually insoluble, the answer being that the completion of a space whose metric takes values in a field IF is one whose metric takes values in the completion of IF. But there are still sufficient problems in making this approach coherent to make one feel that it is simpler to first produce IR from Q, and later repeat the argument when one comes to complete an arbitrary metric space, and of course this destroys the economy of the approach. My own feeling is that in any case the apparatus of Cauchy sequences is logically too complicated for the simple passage from Q to IR one should surely wait until one has the real numbers before doing a piece of analysis! (Conway, 1976, p. 26)

In the above passage Conway (1976) clearly indicates certain care in the teaching of contents that require substantially the notion of real number. Coincidentally, as we have encountered some concerns on the part of Dedekind while teaching the theory of Differential and Integral Calculus, Conway (1976) expresses a similar and necessary warning at the end of his observation. In addition, another element that we seek to emphasize in the next section refers to the fact of how such perspectives, considerations and the tortuous and prolonged mathematical example that provided the evolution of a notion that usually takes an intuitive bias in the school context, requires care and a deep mathematical knowledge on the part of the Mathematics teacher and its specific pedagogy culture.

5. A Brazilian historical research proposal and the Mathematics teacher

In the predecessor sections we address some elements that contribute to the understanding of a historical, mathematical, and evolutionary epistemological process about the notion of real number, whose construction here was accentuated as a contribution by Richard Dedekind, despite the existence of several mathematical methods and strategies for obtain it. On several occasions, we see the influence of Greek thought in the rescue of certain notions worked by Dedekind, as a kind of continuity and systematization of mathematical thinking based in a kind of heuristic perspective. On the other hand, it is worth noting that certain heuristic ideas and, above all, intuitive ideas and reasoning considered by the Greeks were to some extent reconsidered by Dedekind, from a point of view affected by the mathematical theories of his time. In this sense, Conway (1976) attests that:

> Dedekind (and before him the author—thought to be Eudoxus—of the fifth book of Euclid) constructed the real numbers from the rationals. His method was to divide the rationals into two sets L and R in such a way that no number of L was greater than any number of R, and use this "section" to define a new number {L | R] in the case that neither L nor R had an extremal point. (Conway, 1976, p. 3).

Undoubtedly, we can record a series of arguments and strategies taken by Dedekind that were not sufficiently verifies and clarified and from a formal theoretical model. In this sense, we find that the notion of *cut*, the notion of *section* involve certain ideas and presuppositions that by the fact of its intuitive foundation required an extensive period of reflection and systematization of the thought of the old mathematician. Ehrlich (1996) points out fundamental axiomatic properties that were employed by both Dedekind and Cantor´s ideas. Some of these notions, at the time, were taken under consideration of without the necessary care and which, some time later, were the object of analysis and investigation for the work of other mathematicians.

In ours investigations carried out in Brazil on the initial formation of Mathematics teachers, we have sought to emphasize a historical, evolutionary and epistemological understanding regarding notions and mathematical processes, from a non-static point of view and that occurred according to a concern of the mathematicians of the past, however, still has the potential to attract the interest of mathematicians by providing problems that are not completely solved. (Alves, 2018a; 2018b). Ehrlich (1996), for example, comments on certain properties considered by both Cantor and Dedekind that were simply admitted as axioms. Such an attitude must be understood by a mathematics teacher in the sense of understanding the limits of intuition and the formal and structuring theories in Mathematics (Lima, 2000).

> The newly constructed ordered field of real numbers was dubbed the *arithmetic continuum* because it was held that this number system is completely adequate for the analytic representation of all types of continuous phenomena. In accordance with this view, the *geometric linear continuum* was assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to insure this would be the case. In honor of Cantor and Dedekind, who first proposed the thesis, the presumed correspondence between the two structures has come to be called *the Cantor-Dedekind axiom.* Given the Archimedean nature of the real number system, once this axiom is adopted we have the classic result that infinitesimal line segments are superfluous to the analysis of the structure of a continuous straight line. (Ehrlich, 1996, p. viii).

Dedekind made his discovery in 1858 and carried out the actual publication of his theory only in 1872 (DUGAC, 2003, p. 161). Possibly, as we have seen previously, Dedekind himself sought to convince himself of certain facts and properties and that, having regard to its structuralist point of view, could not admit an independent role to the mathematical intuition. Reck (2013) indicates other troubling elements below from the Frege´s criticism.

> This problem is agravated by the fact, also pointed out by G. Frege, that Dedekind does not formulate his basic laws explicitly; much less does he provide a complete list of them. Consequently, it is not clear why Dedekind's theory of 'systems' should be seen as logical. Altogether, it is thus questionable whether a logicist reduction of arithmetic has actually been achieved. (Reck, 2013, p. 142).

We now remember that as general definition we say that a point P that is not represented by any decimal fraction with a finite number n of digits is represented by the infinite decimal fraction

$z.a_1a_2a_3\cdots$, if for every value of n the point P lies in the interval of length 10^{-n} with $z.a_1a_2a_3\cdots$ as its initial point. (Courant & Robbins, 1969, p. 63). In this manner there is established another kind of correspondence between all the points on the number axis and all the finite and infinite decimal fractions We offer the tentative definition: a "number" is a finite or infinite decimal. Those infinite decimals which do not represent rational numbers are called irrational numbers. Courant & Robbins (1969) describe a scenario of scientific effervescence and reformulation of the fundamentals of Mathematics as we can observe below that the role of mathematical intuition persists.

> Until the middle of the nineteenth century these considerations were accepted as a satisfactory explanation of the system of rational and irrational numbers, the continuum of numbers. The enormous advance of mathematics since the seventeenth century, in particular the development of analytic geometry and of the differential and integral calculus, proceeded safely with this concept of the number system as a basis. But during the period of critical re-examination of principles and consolidation of results, it was felt more and more that the concept of irrational number required a more precise analysis. As a preliminary to our account of the modern theory of the number *continuum* we shall discuss in a more or less intuitive fashion the basic concept of limit. (Courant & Robbins, 1969, p. 63)

In any case, despite a critical analysis of several fundamentals of mathematics in the nineteenth century, we find the recurrent recourse to intuition in the sense of delineating certain primitive mathematical thinking, as explained below Courant & Robbins (1969).

> Here the reader may be troubled by an entirely legitimate doubt. What is this "point" on the number axis, which we assumed to belong to all the intervals of a nested sequence, in case it is not a rational point? Our answer is: the existence on the number axis (regarded as a line) of a point contained in every nested sequence of intervals with rational end-points is a fundamental postulate of geometry. No logical reduction of this postulate to other mathematical facts is required. We accept it, just as we accept other axioms or postulates in mathematics, because of its intuitive plausibility and its usefulness in building a consistent system of mathematical thought. (Courant & Robbins, 1969, p. 63)

In Figure 5 we observe another mathematical method that permits determine and define an irrational number. In this case, “an irrational point is completely described by a sequence of nested rational intervals with lengths tending to zero” (Courant & Robbins, 1969, p. 69).

To make this formal definition after having been led to a sequence of nested rational intervals by an intuitive feeling that the irrational point "exists," is to throw away the intuitive crutch with which our reasoning proceeded and to realize that all the matliematical properties of irrational points may be expressed as properties of nested sequences of rational intervals as we can see in the figure!

So far we can see that, despite the mathematical method used to define a real number, intuition is always necessary for the final mathematical reasoning. the mathematics teacher must understand and identify its vestiges that are almost always denied or relativized by mathematicians themselves. Finally, we show another example of an axiom whose heuristic and perceptual meaning does not admit of formal proof.

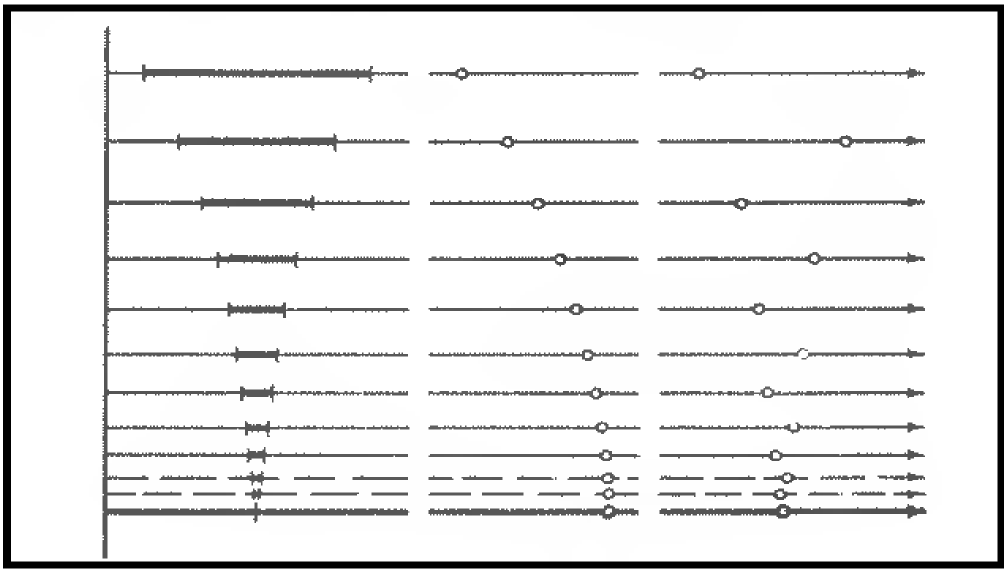

Figure 5. *Courant & Robbins (1969, p. 69) describe an mathematical method for definind a irrational number as defined as nested sequences of rational intervals,*

Axiom: There is a one-to-one correspondence between the points on a Euclidean line r onto the set of real numbers, which is completely determined by the following choices:

(a) A point O on r, to represent the real number 0; (b) A half-line, among those that O determines on r, where we mark the positive reals; (c) A point on the half-line of item (b), to represent the number 1. (Muniz Neto, 2017, p. 16)

On the problem of the lack of continuity of the rational when considering its disposition on the line, Penney (1976, p. 285)) comments that "by virtue of this theorem, it would appear that if the rational umbers were indicated the points on an unbounded straight line, located according to their values as indicated in Fig. 10.2, there could be no gaps in this line". As we have seen in the previous sections, at various times Dedekind suffered with his own intuitions and professional values as a professional mathematician. In this sense, Niven (1956) presents a theorem that can help in the process of refining the mathematical teacher's intuition about real numbers.

Theorem: Almost of real numbes are irrational. (Niven, 1956, p. 2)

Proof. We suggest to the reader appreciate the entire proof in Niven (1956).

Still under the influence of Dedekind's work, we can not desconsidered certain philosophical aspects. In fact, when one defines irrational numbers one assume the existence of rational numbers. But who guarantees the existence of rational numbers? Rations exist and are a consequence of the quotient operation between two integers. But, similarly, who guarantees the existence of whole numbers? But according to Leopold Kronecker (1823 - 1891) the whole numbers are a work of God.

> The language that we use to describe the various classes of numbers is part of our historical inheritance, and so it is not likely to change even though we may feel that some of the words are slightly peculiar. For example, in everyday speech when we describe something as "irrational," we usually mean that it is detached from good sense and therefore unreasonable. But of course we do not regard irrational numbers as being unreasonable. Apparently, the Greeks were surprised when they discovered irrational numbers, because they had felt that, given any two straight-line segments such as the side and the diagonal of a square, there would be some integers a and b so that the ratio of the lengths of the segments would be ajb. Thus the word "rational" in its mathematical sense has reference to this ratio of whole numbers and "irrational" refers to the absence of any such ratio. (Niven, 1961, p. 45).

"In contrast to the rational numbers which were shown to be closed under addition, subtraction, multiplication, and division (except by zero), the irrational numbers possess none of these properties". (Niven, 1961, p. 52). In the above passage, Niven (1961) provides an understanding of the evolutionary epistemological process in Mathematics that never proves to be static and requires constant vigilance on the part of the Mathematics teacher.

6. Conclusion

In the predecessor sections we discuss some elements that allow us to understand some possible trajectories aiming the foundation and axiomatization for the notion of real numbers. We find in the course of the work that the possibilities to determine and construct an ordered and complete (arquimedian) body are not unique, although, according to the method used, certain elements of historical and epistemological order need to be observed and understood by the mathematics teacher.

Dedekind's approach to solving problems related to numbers by finding adequate, very general, and abstract concepts based on choosing certain sets of numbers was *not* easily accepted among mathematicians in the last third of the nineteenth century. (Corry, 2015, p. 230). Moreover, Corry (2015) explains the change of perspective and meaning of the ordered pair, and that at each stage of the construction stage it acquires a distinct meaning and completely loses its meaning in the determining step for the construction of the real. Corry (2015, p. 231) comments that "new concepts and mathematical objects—he said in the talk—continually appear as part of this process, but they must always arise in a natural way from the current state of mathematical knowledge at a given point in time". Here we observe some Dedekind´s ideas about nature of the evolutionary process of mathematical concepts and ideas. This perspective contributes to our understanding of historical research developed in Brazil.

$$? \longrightarrow \mathbb{N} \xrightarrow{(m,n)} \mathbb{Z} \xrightarrow{(p,q)} \mathbb{Q} \xrightarrow{??} \mathbb{R} \xrightarrow{(a,b)} \mathbb{C}$$

$$(m,n) \Leftrightarrow m-n, \quad (p,q) \Leftrightarrow \frac{p}{q}\,(q \neq 0), \quad (a,b) \Leftrightarrow a+bi.$$

Figure 6. *Corry (2015, p. 234) explains the change of meaning of the ordered pairs until arriving at the real numbers.*

An important element, when we consider an initial formation of Mathematics teachers in Brazil, refers to the necessity of transmitting to it a mathematical culture regarding the character of accuracy and precision of Mathematics, but, however, such a current aspect of Mathematics can not to cover up a whole previous scenario of research and discovery by professional mathematicians themselves, who sometimes relied on local heuristics, unspoken thinking and little explanation at the time of their mathematical ideas (Arcavi; Bruckheimer; Ben-Zvi, 1987)

In this context, Niven (1961) observes:

> The system of real numbers, rational and irrational, can be approached at any one of several levels of rigor. (The word "rigor" is used technically in mathematics to denote the degree to which a topic is developed from a careful logical standpoint, as contrasted with a more intuitive position wherein assertions are accepted as correct because they appear somewhat reasonable or self-evident.) (Niven, 1961, p. 7).

The abstract notion of number (Borel, 1952), whether natural, integer, rational or real, usually receives heuristic and intuitive treatment in the school context in Brazil. On the other hand, when we hold the professor of Mathematics, it becomes essential the whole evolutionary, historical and epistemological understanding about his methods of axiomatization and structuring. Such an understanding should constitute an action for the teaching of Mathematics that holds the consciousness of certain structural elements of the axiomatic theory of numbers that need to be omitted in the Brazilian school context.

Throughout the work, we have tried to show that on several occasions the choices and perspective of Richard Dedekind's analysis were not only assumed through an axiolagical, logical and formal apparatus, nevertheless, his personal feeling and his experience as a teacher influenced the necessary and fundamental reflection en virtue the justification of the notion of the real number, although not completely formalized by Dedekind, as we indicated from the criticism of other mathematicians.

Finally, in spite of the recognition of various forms and methods for the formal construction of real numbers, some considered by mathematicians themselves as more general and abstract, while others considered under an undeniably intuitive vies, it is up to the mathematics teacher to understand, distinguish, appreciate and to seek to convey to his students the proficuity and fertility of mathematical reasoning, the inventiveness of mathematicians of the past and, in our special chaos, of the genius manifested by R Dedekind who, despite assuming certain heuristic arguments, the present advance of Mathematics confers that his choices and options, albeit tentative and unexplained, can not be reduced or eliminated.

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