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# COMPARISON OF HIGH-ACHIEVING SIXTH GRADE STUDENTS' PERFORMANCES ON WRITTEN COMPUTATION, SYMBOLIC REPRESENTATION, AND PICTORIAL REPRESENTATION TESTS[1]

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**Abstract:** The present study was conducted to compare the performances of high-achieving sixth grade students on written computation, symbolic representation, and pictorial representation tests. The study enrolled 107 sixth-grade students in three schools. Data were analyzed by using the Kruskal Wallis and Mann Whitney-U Tests and interviews were conducted with six students. The Mann Whitney U test was utilized to explore the groups that were favored by the difference. Student performances were lower particularly in the pictorial representation test than the other two. It is interesting that the students with mathematics grade average points of 4 and 5 at the end of the first academic term failed to show the same success that they showed in the written computation test in the two other tests as well. Therefore, it can be said that the teachers evaluate the students who are good at doing operations as successful in mathematics.

**Key words**: High-achieving sixth grade students, written computation, symbolic representation, pictorial representation, test

## 1. Introduction

Effective computational skills in a student, in other words, his/her skills in performing operations quickly and accurately by using mathematical rules and algorithms, do not necessarily mean that the student has learned the relevant mathematical concept in a meaningful way (Fan & Bokhove, 2014; NCTM, 2000; Zeeuw, Craig & Hye, 2013). Yang and Wu (2010) stated that excessive dependence on rules and algorithms in mathematics teaching reduces students' mathematical thinking skills and prevents their conceptual learning. Many researchers have reported the need for the use of different assessment tools to demonstrate the mathematical performance of students (Cai, 2001; NCTM, 2000).
Representation is among the mathematical process standards laid out in the principles and standards of school mathematics (NCTM, 2000). Students with the skills in translating among multiple representations have a deeper understanding of mathematical ideas. On the contrary, it is difficult to say that students with poor skills in translating among multiple representations have a deep and conceptual understanding (Meij van der & Jong de, 2006). Using multiple representations flexibly is a sign that students have meaningfully learned the mathematical concepts and the relationships among them (Brenner, Herman., & Zimmer, 1999; Debrenti, 2013; NCTM, 2000). Students' skills in translating among multiple representations is crucial for mathematics learning and problem-solving (Acevedo Nistal, van Dooren, Clareboot, Elen & Verschaffel, 2009; Fennell & Rowan, 2001; Gagatsis & Shiakalli, 2004; Pape & Tchoshanov, 2001). When they solve problems, students ought to be able to choose and apply the right forms of representation and possess the skills to translate among multiple representations.

Studies have demonstrated that high-level mathematical thinking and problem-solving strategies may be developed through the flexible use of multiple representations as well as a deep comprehension of

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mathematical concepts (Akkuş, 2004; Cramer, Post & delMas, 2002; Hines, 2002; Sert, 2007). Indeed, many previous studies have revealed a significant relationship between skills in mathematical problem-solving and translating among multiple representations in mathematics teaching (Hitt, 1998; Owens & Clements, 1998). The skills of translating among multiple representations are considered to be a crucial process standard for students in understanding mathematical concepts as well as the relationships between them. Presenting mathematical knowledge in different forms leads to meaningful learning (Ainsworth, 1999, p. 148-149; Akkoç, 2005; İncikabı, Biber, Takıcak, & Bayam, 2015). Indeed, certain earlier studies have mentioned the favorable effects of teaching through multiple representations on students' conceptual learning (Ainsworth, 2006; Rau, Aleven & Rummel, 2009; Schnotz & Bannert, 2003).

In Turkey, curricula and schooling are centralized. All educational institutions operate under the auspices of the Ministry of National Education (MNE), which makes important curricular decisions such as the appointment of teachers or the selection of textbooks and curricular topics. Hence, a national mathematics curriculum is followed in every school (Kurt & Çakıroğlu, 2009, p. 405). One of the specific objectives of the mathematics curriculum in Turkey is: "students should be able to express mathematical concepts in different forms of representation". Despite this, international tests to date have shown that Turkish students display a low level of achievement in the questions which aim to measure their skills of translating among multiple representations. The present study was conducted to compare the performances of students with high achievement in mathematics (students with mathematics grade point averages of 4 and 5 at the end of the first academic term) in the written computation, symbolic representation, and pictorial representation tests. These tests consist of questions about fractions and fractions-related subjects. Also, interviews with students resulted in information about the kind of mathematical problems their teachers posed in class, and what methods they used to solve these problems. Therefore, the present study will reveal the performances of the students with high achievement in mathematics in the written computation, symbolic representation and pictorial representation tests and will also give information about the teachers' instructional styles and how they evaluate student success.

### 1.1. Why fractions and fractions-related subjects?

The concept of fraction is one of the most intangible, complex, and difficult topics that children are supposed to learn at elementary school (Bulgar, 2003; p. 319; Gregg & Gregg, 2007, p. 490; Poon & Lewis, 2007, p. 180; Saxe, Taylor, McIntosh, & Gearhart, 2005, p. 155). Secondary school students experience various difficulties in the fractions subject. The results of the NAEP test show that students' understanding of the fraction concept is rather poor (Sowder & Wearne, 2006, p. 288). It is of utmost importance for students to understand fractions because fractions are associated with natural numbers and operations with natural numbers, ratio, slope, decimal fraction and operations with decimal fractions, percentage and algebra (Brown & Quinn, 2007; Son, 2011). One of the reasons why students experience difficulties is that they transfer their knowledge about natural numbers to fractions (Lamon, 1999; Pitkethly & Hunting, 1996; Sophian, Garyantes & Chang, 1997; Streefland, 1991). Students often seem to be influenced by the presence of natural numbers when they compare the numerical values of fractions (e.g., to believe that 1/4 > 1/3 because 4 > 3) (Obersteiner, Dooren, Hoof & Verschaffel, 2013, p. 64). Another example is related to the comparison of 1/2 and 1/3. As numbers, 1/2 = 0.5 is greater than 1/3=0.(3). But when we consider them as fractions of some quantities, it may only be stated that 1/2 is greater than 1/3 when the reference wholes have the same size (Van de Walle, Karp & Bay-Williams, 2013, p. 296) (Figure 1).

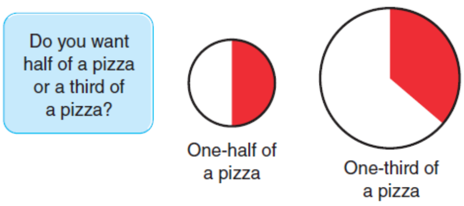



**Figure 1**. *1/2 and 1/3 in wholes of different sizes*

Furthermore, since in the multiplication of natural numbers the product is always greater than the multiplicand and the multiplier, and in the division of natural numbers the quotient is always less than the dividend, students may erroneously tend to transfer this knowledge to fractions. Another reason is the teaching of the algorithms used in fraction operations as a rule, and the failure to help students develop meaningful learning of what these algorithms mean and why they are used through different representations (Van de Walle et al, 2013, p. 315, 330, 333).

As a matter of fact, even some high school students make mistakes such as $(a/b)+(c/d)=(a+c)/(b+d);(a/b):(c/d)=(ad:bc)/bd$ (Li, Chen & An, 2009, p. 811). Another difficulty experienced by the students regarding the subject of fractions stems from the emphasis only on the part-whole interpretation of fractions and the use of pictorial representations that reveal only this interpretation (only the area models). As a result, only 66% of 12-year-olds and only 63% of 13-year-olds responded correctly to the question "Three bars of chocolate are equally distributed to five children. How much chocolate does each child get?" Moreover, fewer students could express that the other representation of 3:5 is 3/5 (Poon & Lewis, 2007, p. 180).

### 1.2. The Role of Representations in Fraction

Previous research has shown that students have a poor performance in using multiple representations and making translations among them. Ni (2001) reported that the factor of representation type significantly affected student performance in finding an adequate representation of a fraction in symbolic form. He added that the best child performance was displayed in tasks involving region models, followed by the discrete objects and finally the number line models. Larson (1988) also offered similar findings. He stated that students were less adept in using fractional terms for discrete object models than for area models. Kara and İncikabı (2018), Kurt and Çakıroğlu (2009) reported that students displayed a surprising deficiency of comprehension in using multiple representations of fractions. According to them, items involving number line models were most problematic.

Representation plays an important role when students are learning about fractions. "Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others" (Bosse, Davis, Gyamfi & Chandler, 2016, p. 1; Gningue, 2016, p. 5-6; Misquitta, 2011; NCTM 2000, p. 67; Tirosh, 2000). It may sometimes be wise to use two different representations in the same activity and require students to make ties between the two alternatives. Researchers seem to agree that a collective use of area, length and set models is important when teaching fractions (Clarke, Roche, & Mitchell, 2008; Siebert & Gaskin, 2006). Naturally, different representations give students different opportunities when learning topics. To illustrate, an area model usually helps students to visualize parts of a whole. On the other hand, the number line not only emphasizes that a fraction is a number, but it also enables comparisons to other numbers, which may not be as clear when area models are used (Van de Walle et al, 2013, p. 294). A linear model is testimony that there is always a fraction to be found between any two numbers, a point which usually goes unnoticed when teaching fractions (Cramer & Whitney, 2010). It is therefore obvious that using different representations and categories of models expands and deepens students' (and teachers') understanding of fractions. Concrete models are necessary in scaffolding students' understanding of, and operations with, fractions. Pictures, contexts, students' language, and symbols are other noteworthy representations. When students can translate among these representations, ideas become meaningful to them (Cramer, Wyberg & Leavitt, 2008, p. 490). It has previously been argued that students who are able to move between visual, verbal and symbolic representations also comprehend the issue better than others (Siegler & Pyke, 2013, p. 1994-1995). Therefore, it may be concluded that moving between different representations supports student learning as well the retention of such learning (Van de Walle et al, 2013, p. 299). Post, Wachsmuth, Lesh, and Behr (1985) assoicated fourth-graders' comprehension of fractions with the flexibility of thought in performing translations between and transformations within modes of representations in rational numbers. Cramer, Post, and delMas (2002) studied 1,600 fourth and fifth graders, and found that statistically higher post test and retention test mean scores were obtained by students in an initial fraction learning program emphasizing the use of and translation among pictorial, manipulative, verbal, real world, and symbolic modes of representation than control students who underwent a regular commercial program. Cramer et al. (2002) state that students' difficulties with learning about fractions are related in part to teaching practices that emphasize syntactic knowledge (rules) over semantic knowledge (meaning). They strongly believe that conceptual understanding should be developed before computational fluency (p.112). Parallel to the aforementioned importance of fractions, the questions used by the data collection tools in the study are associated with the subject of fractions and other relevant mathematics subjects.

### 1.3. Classifications of Multiple Representations

There are different approaches to the classification of representations (Cai, 2005; Goldin 1998; Goldin & Janvier, 1998; Goldin & Shteingold, 2001; Hebert & Powell, 2016; Lesh et al, 1987; Miller & Hudson, 2006; Ponte & Serrazina, 2007, p. 4017; Villegas, Castro & Gutierrez, 2009). According to Ponte and Serrazina (2007), the main forms of representation used in primary education are: "the oral and written language"; "symbolic representations" such as numbers or the signs of the four operations and the equal sign; "iconic representations" such as figures or graphics; and "active representations" such as manipulative materials or other objects. Goldin and Steingold (2001) contend that there are two different systems of representation known by the names external and internal representation systems. For example, children may initially visualize the number '5' in their minds. Then, they can compare the image of the number '5' in their minds and other data sets in the form of "more" or "less." External representations refer to symbols, schema, diagrams, and signs. In other words, an external representation is the equivalent of the thoughts shaped in the mind of the individual in the external world. Villegas et al. (2009) write about the following types of external representations: "verbal representation, which is fundamentally expressed in writing or speech", "pictorial representation, which includes images, diagrams or graphs, as well as certain interrelated activities", "symbolic representation consisting of numbers, operations and connection signs, algebraic symbols and some interconnected actions". Miller and Hudson (2006), divided representation into three types: "concrete", "representational" and "abstract". This taxonomy is similar to that of Bruner in Hebert & Powell (2016), namely "enactive", "iconic" and "symbolic".

Another classification was made by Lesh, Post, and Behr (1987). Lesh et al. divided mathematical representations into five categories: "real-world situation," "concrete models," "arithmetic symbols," "oral or verbal language," and "diagrams or graphs". Among these, the last three reflect the more abstract and higher levels in the representation of mathematical problem-solving (Milrad, 2002; Johnson, 1998; Zhang, 1997). Therefore, it is crucial to have symbolic and pictorial representation skills for problem-solving. Pictorial representation skill refers to the skill in transforming math problems into pictorial form, while symbolic representation skill denotes the skill in expressing mathematical problems in symbolic forms. Signs, expressions, or symbols are symbolic representations and graphs and diagrams are pictorial representations. For example, while "-5" is a symbolic representation, the representation of "-5" in the number line is a pictorial representation. Similarly, while y=3x is a symbolic representation, the linear graph of 3x is a pictorial representation. Figure 2 exemplifies the representation of the fraction 1/3 in different forms. The pictorial representation used circle, length and set models. On the other hand, an example of the symbolic representation of 1/3 may be (1/3=3/9 or 1/3=(4-3)/3).

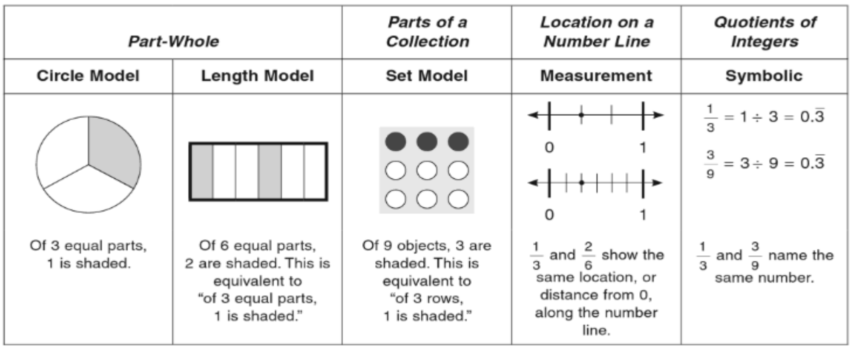



**Figure 2.** *Examples for representation models of fractions*

## 2. Method

A survey was used to explore high-achieving sixth graders' performances in written computation, symbolic representation, and pictorial representation tests. A survey study is defined as "the collection of information from a sample of individuals through their responses to questions" (Check & Schutt, 2012, p. 160). A survey has several characteristics and claimed benefits. Typically, it is used to explore a broad field of issues, populations or programs so as to be able to measure or define generalized features (Cohen, Manion & Morrison, 2007, p. 206).

### 2.1. Participants

The study included 107 sixth-grade students in three schools (nine sixth-grade classes) with the highest achievement in the province-wide tests held in the centre of Yozgat province. Of these students, 56 were girls and 51 were boys. As well as being able to investigate past phenomena, the survey method might also enable researchers to use nonprobability sampling methods, such as purposive sampling (DePoy & Gitlin, 2011). Purposive sampling is preferred when researchers wish to focus particularly on a specific type of person, such as students whose mathematics grades are 4 or 5 out of 5 (Beins & McCarthy, 2011). The calculation of a Turkish sixth-grade student's mathematics grade point average starts with finding the arithmetic means of the students' performance task and class performance grades. The resulting performance grade is then averaged with grades from three written tests and project grade. This yields the student's mathematics achievement point. If this point is between 70-84, the student's grade point average is 4. If it is between 85-100, the grade point average is 5. In this study, students with mathematics grade point averages 4 and 5 were labelled high achievers in mathematics. Even though no consensus exists among researchers regarding the minimum number of participants required in surveys, Cohen, Manion and Morrison (2013) suggest 100 participants minimum. Therefore, the number of participants in the present study can be rendered satisfactory.

### 2.2. Data Collection Tools

The data collection tools used in the study were the "written computation test", "pictorial representation test" and "symbolic representation test" developed by Yang and Huang (2004). First, the items in these tests were translated into Turkish. Following this, the translations were checked by two instructors of mathematics with advanced English levels, and their spelling and intelligibility were checked by two instructors and two teachers of the Turkish language.

Initially, there were 16 items in the tests. Compliance of the test items to the content and objectives of the 6th-grade mathematics curriculum was assessed by seven mathematics teachers, including the mathematics teachers in the selected schools and two instructors with expertise in the field. As a result of the evaluations, the fifth question (Figure 3) in the pictorial representation test was excluded from the test as the subject of "areas of geometric shapes" was not yet covered.

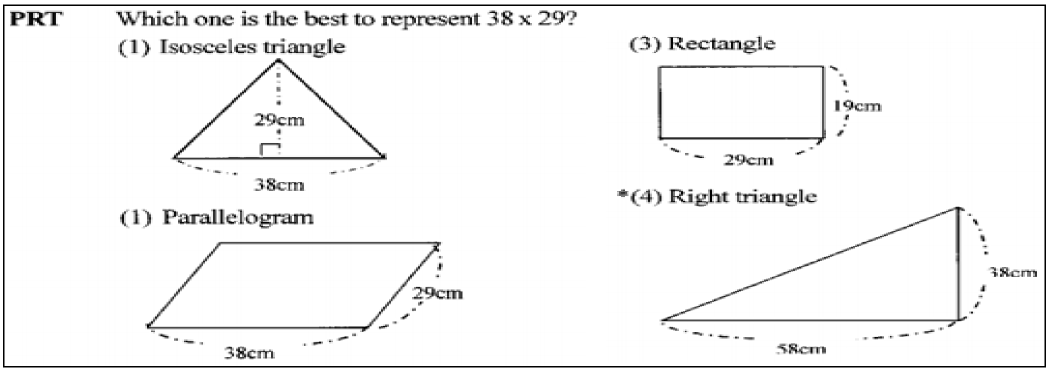



**Figure 3**. *The fifth question in the pictorial representation test*

The fifth questions in the other tests were also excluded from the respective tests as they were considered the equivalents of this question. The data collection tools were piloted with 38 sixth-grade students with similar characteristics (whose mathematics grade point averages were 4 and 5) to the study group. Each test was held during mathematics classes and on different days. To determine whether the students failed to understand any of the questions, a statement saying “Please indicate if there is a statement in the test you cannot understand” was written under each test. None of the students indicated any questions that they failed to understand. It took about one class period (40 minutes) for all of the students to complete the tests. It was observed that the students had more difficulty in solving the questions in the pictorial representation test and that they were faster in finishing the computation test. In each test, the numbers used in the corresponding questions were the same, but they were presented in different question forms. This situation is exemplified below based on the sixth question in the tests (Figure 4).

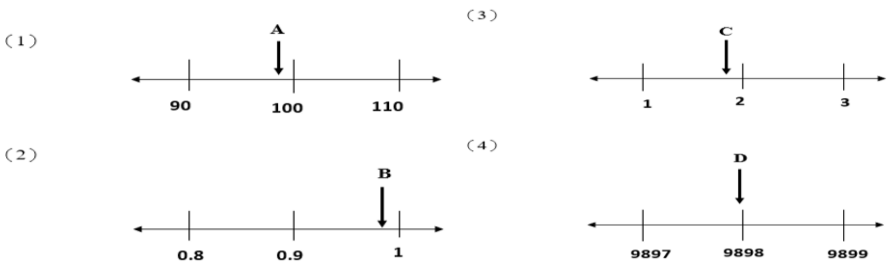
| | |
|---|---|
| WCT | $0.98+\frac{98}{100}=?$ |
| PRT | <br>Since A, B, C, and D are dots on the number line, which of the following is one of the best representations of the result of the mathematical operation 0.98+(98/100)=? |
| SRT | Which of the following can be used in the writing of 0.98+(98/100)<br>(1) $1-0.02+1-\frac{2}{100}$ (2) $98+0.98$ (3) $\frac{0.98+98}{100}$ (4) $0.9+0.8+90+8$ |

**Figure 4.** *Sixth question in the tests*

## 2.3. Implementation

For the validity and internal consistency of the study, each test was implemented in the classes in different weeks. As can be seen in the schedule below (Table 1), the tests were not given after one another in order to avoid distraction and boredom on the students’ part. The tests were also implemented during the first and second class hours in the day (8.00-9.30 a.m.) when students are at their most energetic. The researcher stayed in the classroom during the tests and made necessary explanations.

**Table 1.** *Application schedule of tests*

| Number of class | Date | Test type | Time |
|---|---|---|---|
| 6 | 2019 March 1st week | WRT | First class hour |
| 3 | 2019 March 2nd week | WRT | Second class hour |
| 4 | 2019 March 3rd week | PRT | Second class hour |
| 5 | 2019 March 4th week | PRT | First class hour |
| 5 | 2019 April 1st week | SRT | Second class hour |
| 4 | 2019 April 2nd week | SRT | First class hour |

After the implementation of the tests, interviews were conducted with two students from each school (6 students) with a mathematics grade point average of 5. These students were asked the following three questions: "Which test was more difficult for you and why?"; "What kind of questions do you solve in your math lessons?" and "Which test questions were more similar to the questions you were asked in your math tests and why?". The quantitative findings were supported by qualitative findings from interviews with the students which are given in excerpts after the quantitative findings.

### 2.4. Data Analysis

In each test, the correct answers were scored as 1, and the incorrect or blank answers were scored as 0. Firstly, item analysis was performed on three tests and then principal component factor analysis was applied to show that the tests are congeneric (Yang and Huang 2004, p. 379). The Kruskal-Wallis H test was utilized to explore the presence of statistically significant differences between mean ranks. The analysis indicated a significant difference between the test score mean ranks of at least two groups. For this reason, the Mann-Whitney U test was used to decide whether a significant difference existed between the mean ranks of tests (WRT-SRT; WRT-PRT; SRT-PRT). To reveal the performances of the students in each test more clearly, the number of correct and incorrect answers for each item in the tests, the mean scores from the tests, and the correct answer percentages were given. Descriptive analysis was performed on the findings from interviews with the students, and the quantitative findings were supported by qualitative findings through excerpts from the interviews with the students.

## 3. Findings

### 3.1. Mean and median scores, correct response percentages and standard deviations of the three tests

Table 2 presents the mean scores, correct response percentages and standard deviations of the four tests for sixth-graders. The table shows a much higher performance on the written computation test (WCT) than on the pictorial (PRT) and symbolic representation tests (SRT). These sixth-graders scored best on the WCT, with mean score and correct response percentages at 11.72 and 78.19%, respectively. The PRT ranked the lowest, with mean score and correct response percentages of 6.20 and 40.87%.

**Table 2.** *Mean and median scores, correct response percentages and standard deviations in the three tests*

| Test Types | Mean Scores | Median Scores | Correct response percentages | Standard deviation |
|---|---|---|---|---|
| WCR | 11.72 | 12 | 78.19 | 2.97 |
| SRT | 8.00 | 8.00 | 53.32 | 3.12 |
| PRT | 6.20 | 6.00 | 40.87 | 3.43 |

### 3.2. Factor analysis and item analysis

Factor analysis was employed to reveal the construct validity of the tests. The results of the Kaiser-Meyer-Olkin (KMO) and Bartlett Sphericity tests were examined to evaluate the adequacy of the sample and the appropriateness of the data for factor analysis (Field, 2005). According to Hair, Black and Babin (2010); Pallant, (2007); Tabachnick and Fidell (2007), the Kaiser-Meyer-Olkin (KMO) value must be greater than .60 and the Bartlett's Test of Sphericity (BTS) must be significant at $\alpha <$ .05 for exploratory factor analysis.

**Table 3.** *KMO and Bartlett's test values*

| | | |
|---|---|---|
| Kaiser-Meyer-Olkin Sampling Adequacy | 0.655 | |
| Bartlett Sphericity Test | $\chi^2$ | 110.411 |
| | df | 3 |
| | p | 0.000 |

Table 3 shows that factorizing the three tests (KMO= .66>.60 and Bartlett's Test of Sphericity significant at .05) is justified (Hair, Black & Babin, 2010; Pallant, 2007; Tabachnick & Fidell, 2007). The summary statistics from the principal components analysis in Table 4 reveal that one dominant factor underlies all three tests as the first component accounts for more than 71.78% of variance ($\lambda$=2.15) and the eigenvalue for the second component is below 1. This suggests that the three tests are congeneric tests, all tapping into the same aspect (Yang & Huang, 2004, p.379).

**Table 4.** *Eigenvalues, explained variance % and cumulative proportion of total variance from principal component analysis on the three tests*

| Dimension | Eigenvalue | Variation | Cumulative variation |
|---|---|---|---|
| 1 | 2.154 | 71.788 | 71.788 |
| 2 | 0.558 | 18.603 | 90.391 |
| 3 | 0.288 | 9.609 | 100.00 |

To determine the internal consistency of the tests, item analysis was performed, and KR-20 coefficients were examined. Even though KR-20 and alpha coefficients imply the same, the name KR-20 is reserved for dichotomously scored items, while alpha is reserved for polytomously scored ones. "Alpha is a general version of the Kuder-Richardson coefficient of equivalence. It is a general version because the Kuder-Richardson coefficient applies only to dichotomous items, whereas alpha applies to any set of items regardless of the response scale" (Cortina, 1993, p. 99). The analyses showed that the discrimination index of all items were $\geq 0.20$ (Appendix 1). The KR-20 coefficient was found to be .76 for the pictorial representation test, .72 for the symbolic representation test, and .74 for the written computation test. Generally, KR-20 coefficients above .70 indicate an acceptable level (Erwin, 2000). These results show that the tests are reliable.

### 3.3. Analysis of differences among the WCT, PRT and SRT

The students performed better in each question in the written computation test than in the pictorial representation test. Similarly, students' performances in the written computation test were found to be higher than their performances in the symbolic representation test. A contrary result was obtained only in the 15th question. When the performances of the students in the symbolic representation and pictorial representation tests were compared, in general, the students performed better in the symbolic representation test. However, a contrary result was obtained in the 4th, 5th, 9th, 12th, and 13th questions. To better demonstrate the differences between the performances of the students in the tests, the answers given to questions 9, 11, 14 and 15 and the percentage of the alternatives chosen by the students are examined below (Table 5).

**Table 5.** *Number of correct and incorrect answers and correct answer percentage in the tests*

| Question | Test Type | | | | | |
|---|---|---|---|---|---|---|
| | WCT | | SRT | | PRT | |
| | Correct (%) | Incorrect (%) | Correct(%) | Incorrect (%) | Correct (%) | Incorrect(%) |
| 1 | 89 (83.2) | 18 (16.8) | 84 (78.5) | 23 (21.5) | 68(63.6) | 39 (36.4) |
| 2 | 86 (80.4) | 21 (19.6) | 81 (75.7) | 26 (24.3) | 33 (30.8) | 74 (69.2) |
| 3 | 87 (81.3) | 20 (18.7) | 55 (51.4) | 52 (48.6) | 52 (48.6) | 55 (51.4) |
| 4 | 59 (55.1) | 48 (44.9) | 15 (14.0) | 92 (86.0) | 56 (52.3) | 51 (47.7) |
| 5 | 88 (82.2) | 19 (17.8) | 18 (16.8) | 89 (83.2) | 42 (39.3) | 65 (60.7) |
| 6 | 88 (82.2) | 19 (17.8) | 34 (31.8) | 73 (68.2) | 32 (29.9) | 75 (70.1) |
| 7 | 102 (95.3) | 5 (4.7) | 89 (83.2) | 18 (16.8) | 67 (62.6) | 40 (37.4) |
| 8 | 77 (72.0) | 30 (28.0) | 68 (63.6) | 39 (36.4) | 30 (28.0) | 77 (72.0) |
| 9 | 100 (93.5) | 7 (6.5) | 21 (19.6) | 86 (80.4) | 23 (21.5) | 84 (78.5) |
| 10 | 96 (89.7) | 11 (10.3) | 73 (68.2) | 34 (31.8) | 69 (64.5) | 38 (35.5) |
| 11 | 97 (90.7) | 10 (9.3) | 80 (74.8) | 27 (25.2) | 27 (25.2) | 80 (74.8) |
| 12 | 63 (58.9) | 44 (41.1) | 38 (35.5) | 69 (64.5) | 55 (51.4) | 52 (48.6) |
| 13 | 85 (79.4) | 22 (20.6) | 39 (36.4) | 68 (63.6) | 46 (43.0) | 61 (57.0) |
| 14 | 80 (74.8) | 27 (25.2) | 76 (71.0) | 31 (29.0) | 34 (31.8) | 73 (68.2) |
| 15 | 58 (54.2) | 49 (45.8) | 85 (79.4) | 22 (20.6) | 22 (20.6) | 85 (79.4) |

The majority of the students answered the 9$^{th}$ question (1 2/5+ 2 4/5 in WCT) correctly. However, they failed to do so when identifying the symbolic and pictorial representation form of this operation (Appendix 2). Although 93.5% of the students calculated this operation correctly, 31.7% could answer this question correctly in the pictorial form, while only 19.6% could answer it correctly in the symbolic representation form. When the answers of the students were analyzed, it was seen that a considerable number thought that the result of 1 2/5 +2 4/5 is 3 6/10. Therefore, the students made the common mistake in fractions by adding the numerators and writing the result over the denominator and adding the denominators and writing the result under the numerator. 21.4% of the students thought like this in the pictorial representation test, while 41.1% thought like this in the symbolic representation test.

Similarly, 90.7% of the students answered the 11$^{th}$ question (5/7-9/14 in WCT) correctly (Appendix 3). The percentage of students who could answer the equivalent of this question in the symbolic representation test was 74.8, while only 25.2% of the students could answer the equivalent of this question in the pictorial representation test. The apparent mistake made by the students in the 11$^{th}$ question in the pictorial representation test was that they tried to compare 5/7 and 9/14 over wholes of varying sizes. Of the students, 64.4% made this mistake. The most common mistake in the symbolic representation test was that they subtracted the smaller numerator from the larger numerator (9-5) and, similarly, they subtracted the smaller denominator from the larger denominator (14-7). The majority of the students answered the 14$^{th}$ question correctly in WRT and SRT. However, the students were unable to display a similar successful performance when identifying the pictorial representation form of this operation (Appendix 4). Although 80% calculated this operation correctly, 76% could answer this question correctly in the symbolic form, while only 34% could answer it correctly in the pictorial representation form. Considering the performances of the students in the 15$^{th}$ question (0,81-0,799) in each test, the percentage of success in the symbolic representation test was 79.4, in the written computation test 54.2, and in the pictorial representation test only 20.5 (Appendix 5). The most apparent reason for the failure to answer the 15th question in the pictorial representation test was a misconception about decimal numbers. A common misconception that students have about decimals is as follows: "The longer the number, the larger the number" (Karp, Bush, & Dougherty, 2014, p. 23). This shows that students resort to whole number thinking as they examine numbers to the right of a decimal. For instance, a student may incorrectly conclude that 3.175 > 3.4 as they already know that 175 is greater than 4. A related misconception is "the longer the decimal, the smaller the number" (Griffin, 2016). This error happens when students conclude that 2.725 < 2.7 because thousandths is smaller than tenths (Brown, 1981). Failure to include the number line representation in textbooks and courses may be one of the underlying reasons for the poor performance of students in this question. In general, the students performed better in the written computation test than in the others. The students showed the lowest performance in the pictorial representation test. Below are the results of the Kruskal Wallis and Mann-Whitney U tests which were used in order to explore whether students' test performances varied significantly.

### 3.4. The analysis of Kruskal-Wallis and Mann-Whitney U test results

Before conducting one-way ANOVA, assumptions of the ANOVA test were tested (Meyers, Gamst & Guarino, 2013, p.140). The Kolmogorov–Smirnov and Shapiro–Wilk tests are the most widely used methods in testing the normality of data. Although the Shapiro–Wilk test is preferred for small sample sizes ($n < 50$), it can be used with larger sample sizes as well. The Kolmogorov–Smirnov test, on the other hand, is used for ($n >= 50$). For both of the above tests, the null hypothesis states that data are taken from a normally distributed population. When $p > .05$, the null hypothesis was accepted and data were considered normally distributed (Elliott & Woodward, 2007; Mishra, Pandey, Singh, Gupta, Sahu & Keshri, 2019). The p significance level was below 0.05 in this study (Table 6), thus not satisfying the normality assumption (Field, 2013). Therefore, the analysis was conducted by using the Kruskal-Wallis test. Table 7 presents the results of the Kruskal-Wallis analysis for the three tests.

**Table 6.** *Normality for the tests*

| Tests | | Kolmogorov-Smirnov | | |
|---|---|---|---|---|
| | | Statistic | df | Sig. |
| WCT | Score | .17 | 107 | .00 |
| SRT | Score | .09 | 107 | .02 |
| PRT | Score | .13 | 107 | .00 |

**Table 7**. *Findings of the Kruskal Wallis analysis*

| Score | Test Type | N | Mean Rank | $\chi^2$ | df | p |
|---|---|---|---|---|---|---|
| Test Score | WCT | 107 | 233.71 | 109.97 | 2 | .00 |
| | SRT | 107 | 145.50 | | | |
| | PRT | 107 | 103.79 | | | |
| | Total | 321 | | | | |

The analysis showed a significant difference between the mean ranks of at least two groups. The Mann-Whitney U test was employed to explore the tests that had a significant difference in terms of mean ranks. The findings obtained from the analysis are given in Table 8.

**Table 8.** *Findings of the Mann-Whitney U analysis*

| Tests | N | Mean Rank | Rank Total | U | p |
|---|---|---|---|---|---|
| WCT | 107 | 139.97 | 14977.00 | 2250.00 | .00 |
| SRT | 107 | 75.03 | 8028.00 | | |
| **Tests** | | | | | |
| WCT | 107 | 147.74 | 15808.50 | 1418.500 | .00 |
| PRT | 107 | 67.26 | 7196.50 | | |
| **Tests** | | | | | |
| SRT | 107 | 124.47 | 13318.00 | 3909.000 | .00 |
| PRT | 107 | 90.53 | 9687.00 | | |

The Mann-Whitney U test aiming to explore whether a significant difference existed between students' test performances showed that their performances in the written computation and symbolic representation tests varied significantly in favor of the former (U=2250, $p < .05$). Similarly, student performances in the written computation and the pictorial representation tests varied significantly in favor of the former (U=1418.5, $p < .05$), and between the symbolic representation and pictorial representation tests again in favor of the former (U=3909, $p < .05$). When one-way Anova was conducted, the results were the same as for the non-parametric tests. To reveal the possible causes of this result, interviews were conducted with two students from each school (6 students) with a mathematics grade point average of 5. The interviews were analyzed to determine the tests that were the easiest and hardest for the students and their reasons. In addition, the scores obtained by the interviewed students on each test were given to see whether there was a relation between students' views about the difficulty level of the tests and their performance in them. Findings from the analyses of interview data are given in Table 9.

**Table 9**. *Findings from interviews with students*

| Students | The most difficult test for each student | The easiest test for each student | The reason for the student response | Student performance in tests | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | PC | | SR | | WC | |
| | | | | C | I | C | I | C | I |
| 1 | PC | WC | Questions the teacher asked in class and in the exam | 5 | 10 | 7 | 8 | 11 | 4 |
| 2 | PC | WC | | 7 | 8 | 8 | 7 | 14 | 1 |
| 3 | PC | WC | | 6 | 9 | 9 | 6 | 12 | 3 |
| 4 | SR | WC | | 4 | 11 | 8 | 7 | 14 | 1 |
| 5 | PC | WC | | 6 | 9 | 7 | 8 | 12 | 3 |
| 6 | PC | WC | | 5 | 10 | 6 | 9 | 13 | 2 |

C: Number of correct answers; I: Number of incorrect answers

Five out of the 6 interviewed students stated that the most difficult test was the pictorial representation test, while the easiest was the written calculation test. The remaining student said that the symbolic representation test was the hardest. This student agreed with others that the written calculation test was the easiest. The main reasons why the students found pictorial and symbolic representation tests more difficult included the lack of such questions in their classes and the teachers using questions similar to written calculation test questions in their classes and actual tests. This was reflected in the test performance of all six students. Below are some excerpts from the interviews with the students.

*Researcher: Which test was more difficult for you? Can you explain why?*
*Student A: I had more difficulty in the pictorial representation test because I was not familiar with the question types.*
*Researcher: What do you mean you were not familiar?*
*Student A: I mean, we do not solve such questions in math classes.*
*Researcher: Then, what kind of questions do you solve?*
*Student A: Like the ones in the first test. The written computation test. We solve the questions using rules.*
*Researcher: So what kind of questions are asked in your tests?*
*Student A: Generally problems or operations. That's why I was not familiar with the questions in the symbolic representation test and the pictorial representation test.*

*Researcher: Which test was more difficult for you?*
*Student B: The last one you gave was the most difficult one.*
*Researcher: Can you explain why?*
*Student B: It was the first time I encountered such questions; that's why.*
*Researcher: Didn't you ever solve such questions in your classes or tests?*
*Student B: No.*
*Researcher: Then, what kind of questions does your teacher ask in class?*
*Student B: Written computation questions. For example, multiplication, addition, and subtraction with fractions. We have never solved pictorial questions.*

*Researcher: How would you rank the tests from the most difficult to the easiest?*
*Student C: I think the last test is the hardest, and the first is the easiest.*
*Researcher: Why do you think so?*
*Student C: Because we solve similar questions in class. There were symbols in the second test, so the second test was moderately difficult. But I had never encountered the questions asked in the last test.*

Interviews with students showed that they were more familiar with the questions in the written computation test, while the questions in the pictorial representation test were not included in their classes or tests. This finding indicates that teachers do not ask questions to measure students' skills in translating among multiple representations, a skill which teachers fail to emphasize in their classes. As a result, it may be stated that the qualitative findings support the quantitative findings.

## 4. Discussion and Conclusion

The present study was conducted to compare the performances of 107 sixth-grade students with high achievement in mathematics in the written computation, symbolic representation, and pictorial representation tests. The Mann-Whitney U test was performed to determine whether the test performances of the students varied significantly, and it was ascertained that their performances in the written computation and symbolic representation tests varied significantly in favor of the former. A significant difference was also found between the students' performances in the written computation and pictorial representation tests in favor of the former, and between the symbolic representation and pictorial representation tests once again in favor of the former. The median of the students' scores in the written computation test was 12, and the average correct answer percentage was 78.19. The students failed to achieve the same high scores in other tests. The students' performances in the pictorial representation test were particularly low. While the median of the scores from the symbolic representation test was 8 and the average correct answer percentage was 53.32, the median of the scores from the pictorial representation test was 6, and the average correct answer percentage was 40.87. A poorer performance in the pictorial representation test parallels previous results reported in the literature (Kara & İncikabı, 2018; Kurt & Çakır, 2009; Larson, 1988; Ni, 2001). Cramer et al. (2002) state that students' difficulties with learning about fraction are related in part to teaching practices that emphasize syntactic knowledge (rules) over semantic knowledge (meaning). They strongly believe that conceptual understanding should be developed before computational fluency (p.112). Making connections between different representations is critical in developing mathematical understanding. When students internalize the use of representations, they will be better at comprehending and calculating fractions (Charalambous & Pantazi, 2007).

By allowing students to explore fraction operations by first using representations that make sense to them, they come to understand mathematics more deeply (Smith, Bill & Raith, 2018, p. 39). In this respect, mathematics teaching should aim at the simultaneous development of procedural fluency and conceptual understanding in students (Boerst & Schielack, 2003; NCTM, 2000).The results showed a substantial deficiency in understanding on the students' part in using multiple representations of fractions. This finding may be attributed to a lack of emphasis on multiple representations in mathematics classes. Indeed, at the time of the data collection, schools did not seem to place any emphasis on the multiple representations of mathematical concepts. Kurt and Çakıroğlu (2009) observed students to display a substantial lack of understanding in using multiple representations of fractions, contrary to the expectation. They seemed to have the biggest problems in items involving number line models. Kara and İncikabı (2018) investigated the representation preferences of sixth grade students when adding and subtracting fractions and the success rate of their preferences. They found the area model to be the most commonly used type of representation (42%). This was followed by numerical representation and then verbal representation. However, they found that number line and verbal representation were not commonly preferred. An over-reliance on any one representation limits students' conceptual understanding of fractions. Researchers extol the virtues of collective use of area, length and set models when teaching fractions (Clarke, Roche, & Mitchell, 2008; Siebert & Gaskin, 2006). Different representations also present students with different opportunities to learn. To illustrate, students can visualize parts of a whole with the use of an area model. On the other hand, the number line shows that a fraction is one number and it also allows comparisons in its relative size to other numbers, which may not be as clear with area models (Van de Walle et al, 2010).

A linear model draws students' attention to an aspect that is often underestimated in the teaching of fractions; namely, that there is always another fraction to be found between any two numbers (Cramer & Whitney, 2010). For effective learning, students should be allowed to explore fractions across area, length, and set models. If they never see fractions represented as a length, they will naturally have problems in solving linear problems. A teacher cannot be sure that the students have fully understood the meaning of 1/4 if they are not given opportunities to represent one-fourth using area, length, and set models. Therefore, teachers should use different fraction representations (area models, cluster models, linear models, verbal and symbolic expressions) for students to develop a deep learning of fractions and to help them overcome their misconceptions about fractions (Bezerra, Magina & Spinillo, 2002; Van de Walle et al, 2010). Figure 4 shows examples of written, pictorial and symbolic representations that can be used for fractions. The red and white beads in figure 4 represent the proportional meaning of the fraction, while the green and blue circle slices represent the part-whole meaning of the fraction.

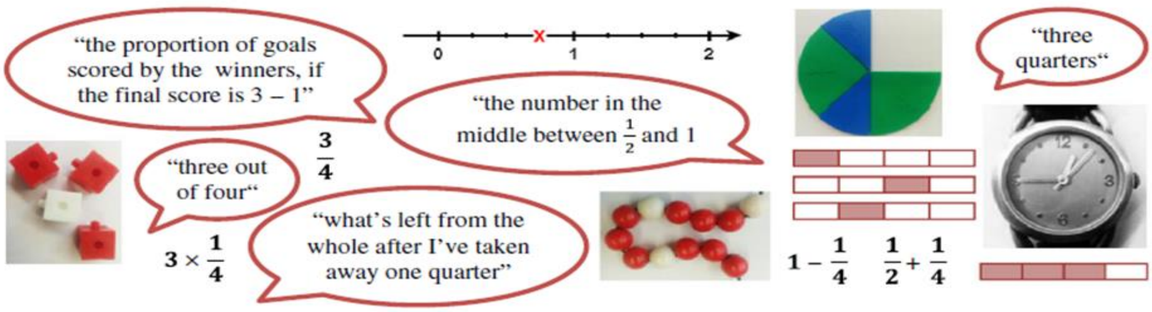



**Figure 4**. *Some representations for a fraction (Dreher, Kuntze & Lerman, 2016).*

It is interesting that the students with mathematics grade average points of 4 and 5 at the end of the first academic term failed to display the success they had in the written computation test in the other two tests as well. It should be noted that these students were considered by their teachers as highly successful in their mathematics lessons. The results may be related to the teaching and assessment approach that the teachers adopt. As a matter of fact, the interviews with students revealed that their teachers ask questions similar to the questions in the written computation test, and they evaluate their students' success based on such questions. Therefore, it is obvious that the teachers evaluate the students who are good at operations as successful in mathematics. Effective computational skills in a student, in other words skills in performing operations quickly and accurately by using mathematical rules and algorithms, do not necessarily mean that the student has learned the relevant mathematical concept in a meaningful way (Fan & Bokhove, 2014; Zeeuw, Craig & Hye, 2013). The present study was conducted to reveal the relationship between the performances of students with high achievement in mathematics in the written computation, symbolic representation, and pictorial representation tests. The study was limited to 107 high achieving students from 3 different schools (9 sixth-grade sections) and 15 questions in 3 tests. Hence the findings may not be generalized to all sixth grade students in Turkey. Future studies may involve more students with different mathematics achievement levels and cover different topics.

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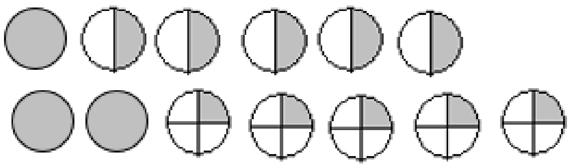
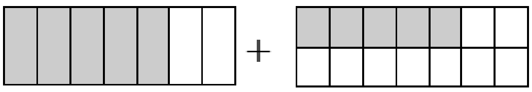
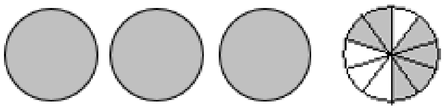
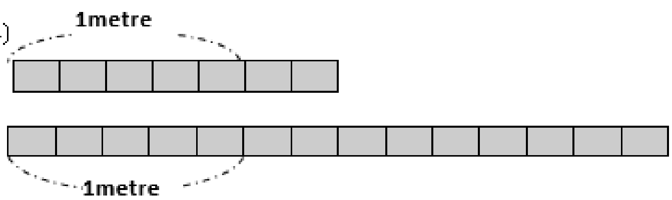
## Appendix

**Appendix 1**. *Item Analysis Findings*

| Item | WCT | | SRT | | PRT | |
|---|---|---|---|---|---|---|
| | DIF | DI | DIF | DI | DIF | DI |
| 1 | .83 | .29 | .78 | .24 | .63 | .39 |
| 2 | .80 | .30 | .75 | .35 | .30 | .31 |
| 3 | .81 | .27 | .51 | .46 | .48 | .40 |
| 4 | .55 | .25 | .14 | .20 | .52 | .37 |
| 5 | .82 | .46 | .16 | .48 | .39 | .54 |
| 6 | .82 | .31 | .31 | .27 | .29 | .55 |
| 7 | .95 | .41 | .83 | .34 | .62 | .23 |
| 8 | .71 | .74 | .63 | .29 | .28 | .32 |
| 9 | .93 | .37 | .19 | .30 | .21 | .44 |
| 10 | .89 | .40 | .68 | .35 | .64 | .40 |
| 11 | .90 | .39 | .74 | .44 | .25 | .49 |
| 12 | .58 | .38 | .35 | .46 | .51 | .53 |
| 13 | .79 | .38 | .36 | .47 | .42 | .62 |
| 14 | .74 | .46 | .71 | .25 | .31 | .32 |
| 15 | .54 | .58 | .79 | .39 | .20 | .20 |
| KR-20 | .74 | | .72 | | .76 | |

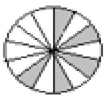
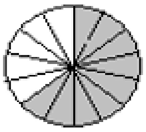
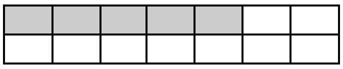
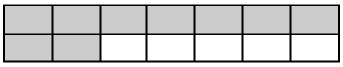
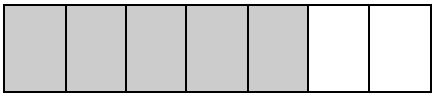
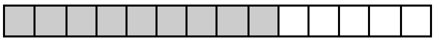
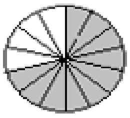
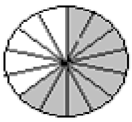
DIF: difficulty index; DI: discrimination index

**Appendix 2**. *Answers to the 9th question in each test*

| **Test** | **Item 9** | **Percentage of Correct Answers** |
|---|---|---|
| Computation | $1\frac{2}{5}+2\frac{4}{5}=$ | (93.5%) |
| Pictorial Representation | $1\frac{2}{5}+2\frac{4}{5}$ which is one of the best representations of the result of the preceding operation?<br>(1) <br>(2) <br>(3) <br>(4)  | (1)(25.2%)<br>(2)(21.4%)<br>(3)(21.4%)<br>(4) (31.7%)* |
| Symbolic Representation | $1\frac{2}{5}+2\frac{4}{5}=$<br>(1) $1+2+\frac{2+4}{5+5}$<br>(2) $1+0.4+2+0.8$<br>(3) $1x\frac{2}{5}+2x\frac{4}{5}$<br>(4) $\frac{2}{5}+\frac{8}{10}$ | (1)(41,1%)<br>(2)(19,6%)*<br>(3)(33,6%)<br>(4) (5.6%) |

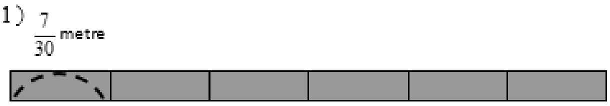
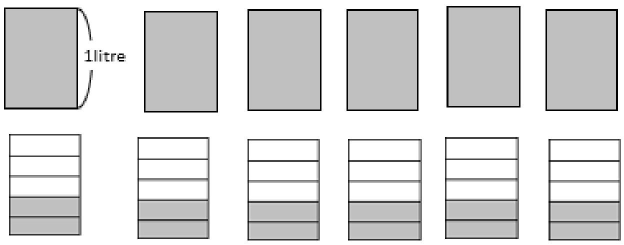
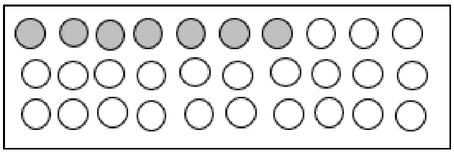
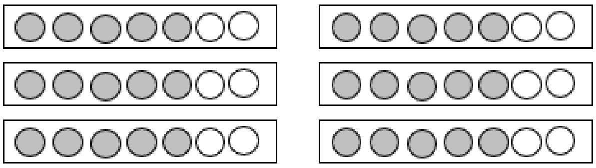
*Correct answer

**Appendix 3**. *Answers to the 11th question in each test*

| **Test** | **Item 11** | **Percentage of Correct Answers** |
|---|---|---|
| Computation | $\frac{5}{7} - \frac{9}{14} =$ | (90.7%) |
| Pictorial Representation | $\frac{5}{7}$ $\frac{9}{14}$ in which alternative is the comparison of the preceding fractions is best represented?<br>(1)<br>$\frac{5}{7}$  $\frac{9}{14}$ <br>(2)<br>$\frac{5}{7}$ <br>$\frac{9}{14}$ <br>(3)<br>$\frac{5}{7}$ <br>$\frac{9}{14}$ <br>(4)<br>$\frac{5}{7}$  $\frac{9}{14}$  | (1) (7.4%)<br>(2) (10.2%)<br>(3) (57%)<br>(4) (25.2)* |
| Symbolic Representation | $\frac{5}{7} - \frac{9}{14} =$<br>(1) $\frac{9-5}{14-7}$<br>(2) $\frac{10-9}{14}$<br>(3) $\frac{9-5}{14}$<br>(4) $57 - 9.14$ | (1) (14.9%)<br>(2) (74.8%)*<br>(3) (7.4%)<br>(4) (2.8%) |

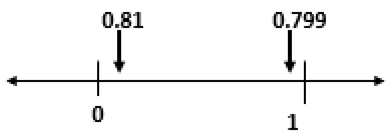
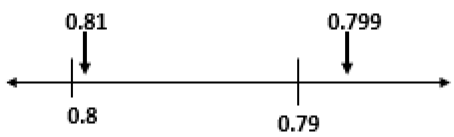
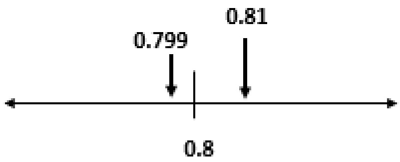
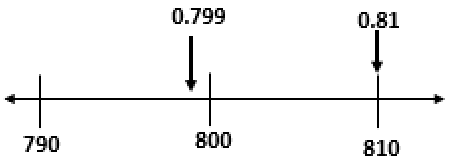
*Correct answer

**Appendix 4.** *Answers to the 14th question in each test*

| **Test** | **Item 14** | **Percentage of Correct Answers** |
|---|---|---|
| Computation | $\frac{7}{5} \times 6 = ?$ | (80%) |
| Pictorial Representation | $\frac{7}{5} \times 6 = ?$ which is one of the best representations of the result of the preceding operation?<br>(1)  <br>(2)  <br>(3) <br>(4)  | (1) (25.2%)<br>(2) (34%)*<br>(3)(31.8%)<br>(4) (9%) |
| Symbolic Representation | $\frac{7}{5} \times 6 = ?$<br>(1) $\frac{7 \times 6}{5}$<br>(2) $\frac{7}{5} \times \frac{6}{6}$<br>(3) 1.2×6<br>(4) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ | (1) (76%)*<br>(2) (12,6%)<br>(3) (5.8%)<br>(4) (5.6%) |

*Correct answer

**Appendix 5**. *Answers to the 15th question in each test*

<table>
<tr><th>Test</th><th>Item 15</th><th>Percentage of Correct Answers</th></tr>
<tr><td>Computation</td><td>$0.81 - 0.799 =$</td><td>58 (54.2%)</td></tr>
<tr><td>Pictorial Representation</td><td>What is one of the best representations of the comparison of 0.81 and 0.799?<br>
(1)<br>

<br>
(2)<br>

<br>
(3)<br>

<br>
(4)<br>

</td><td>(1)(32.7%)<br>(2)(10.2%)<br>(3)(20.5%)*<br>(4)(36.4%)</td></tr>
<tr><td>Symbolic Representation</td><td>Which of the following is correct?<br>
(1) $0.81 + 0.799 = \frac{81}{100} + \frac{799}{1000}$<br>
(2) $0.81 + 0.799 = (81 + 799)x0.001$<br>
(3) $0.81 + 0.799 = (81 - 79.9)x\frac{1}{1000}$<br>
(4) $0.81 + 0.799 = 0.01x(799 - 81)$</td><td>(1)(79.4%)*<br>(2)(11.2%)<br>(3)(0%)<br>(4)(9.3%)</td></tr>
</table>

*Correct answer