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TEACHERS' APPROACHES TO STUDENT ERRORS IN MATHEMATICS TEACHING AND NOTICING OF THEIR APPROACHES

Gülşah ÖZDEMIR BAKI, Merve ÖZKAYA, Alper Cihan KONYALIOĞLU

Abstract: The aim of this study is to examine the error approach of middle school teachers and their noticing of their approaches. The study was designed as a case study, which is a qualitative research method, and conducted through four middle school mathematics teachers during the pandemic. The data regarding the study were gathered via in class video records and online interviews with the teachers. The videos from the lectures of middle school mathematics teachers were segmented and analysed, hence, their approaches to student errors were identified. Moreover, noticing of the participant teachers on their and their peers' approach were investigated by means of online interviews. The findings of the research indicated the participant teachers mostly opt for the intervention approaches that fall into the category of questioning, and they make delineative and explanatory comments concerning their approaches. Furthermore, it has been observed that the teachers reached deductions through the evaluation of their own approaches, however, most of the time they recommended their own error approaches to the students as an educatory proposition.

Key words: Error, Error approach, Intervention, Teacher noticing, Mathematics teacher, Video analysis

1. Introduction

Due to the crucial role of teachers on the education quality, myriad of research material was produced in order to delineate the qualities of a good teacher and a plethora of views have been asserted regarding the issue. (Shulman, 1986; 1987, National Council of Teachers of Mathematics [NCTM], 2000, Charalambous & Hill, 2012; Ball, Thames & Phelps, 2008; Hill, Rowan & Ball, 2005; Ma, 2010). Eventually, these qualities became more pronounced with more definitive components thanks to Shulman (1986;1987) and his fellows. Student knowledge became the center of interest among these subcomponents (Park & Oliver, 2008). The term student knowledge encompasses teachers' noticing of errors that students might possess and being erudite on what knowledge student should have (NCTM, 1989; 1991; 2000). It is important to analyse the student errors when this situation has been given thought (Graeber, 1999). Apart from the analysis of student errors in the classroom, how teachers interested in and intervene the errors must be emphasized (Santagata, 2002). Moreover, teachers must be encouraged to notice their approaches.

Error approach, which is a fundamental pillar of mathematics education, is the most effective method (Pirie & Kieren, 1994) teachers employ to understand students and an invaluable part in improving the quality of education (NCTM, 1989; 1991). Because teachers who adopt the correct approach to student errors can understand what errors students have and analyse the errors better (Shulman, 1986). The ability to analyse student errors is directly pertinent to the conceptual knowledge level (Rach, Ufer & Heinze, 2013). The importance of analysing the student errors is emphasized in the teachers' development of conceptual knowledge level as well (Borasi, 1986; Carpenter, Franke & Levi, 2003; Ma, 2010; Tsamir, 2007). Most of the research that were conducted on error approach mostly focused on teachers' or teacher candidates' conceptual approaches to errors (Chick & Baker, 2005; Baker & Chick, 2006; Peng & Luo, 2009; Son & Crespo, 2009; Demirci, Ozkaya & Konyalioglu, 2017). Nonetheless, there has been a scarcity on the research of error approaches in terms of interventions (Santagata, 2004; Son & Sinclair, 2010; Steuer, Rosentritt-Brunn & Dresel, 2013; Didis, Erbas & Cetinkaya, 2016). Besides the studies on identification of errors and explaining the root causes,

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investigating the approaches including interventions is especially important when the teachers who will teach the posterity are considered. Hence, teachers must be provided with the opportunities to notice their intervention approaches so that they can be more effective in classroom. One of these opportunities are the videos by which they can observe their or their peers' error approaches in classroom (Borko, Jacobs, Eiteljorg & Pittman, 2008; Sherin & van Es, 2009). Teachers can become aware of their error approaches by means of watching their own or their peer's videos of in class applications. Thus, teachers can analyse learning in a better way (Rodgers, 2002).

Within the scope of this study, the intervention approaches of teachers to student errors will be identified. Error approaches will be demonstrated in this context. The noticing of teachers, who watched their own or their peer's videos of in class activities, on their error approaches will be identified. It has been envisioned that the findings of this study will contribute to teachers' learning of better error approaches, growing a pedagogical perspective to the errors and support teachers' occupational development. The videos which possess in class activities open a window to class classroom ambient and orient teachers to comparative and critical thinking (Gaudin & Chaliès, 2015). The significance of the study is evident since it will contribute to other research that investigates the error approaches. Moreover, noticing of the error approaches are essential for the education of teachers as well. Therefore, it has been intended to identify the error approaches of middle school mathematics teachers to student errors and teacher noticing on their approaches. The study was evaluated in two dimensions. The first dimension is student error approach, and the other is teacher noticing. Under the light of the purposes, the answers were sought for the following research questions below:

1. How are the error approaches of middle school mathematics teachers to student errors in terms of interventions?
2. How are the noticing of middle school mathematics teachers on their approaches to student errors in terms of interventions?

2. Theoretical Framework

2. 1. Error Approach

It has been seen that there exists a negative perspective to errors prior to the constructive approach (Borasi, 1996). Error became an educational activity instead of a concept that must certainly be avoided (Gartmeier, Bauer, Gruber & Heid, 2008; Tauber, 2009) along with the constructive approach (Borasi, 1996). Errors are not just education activities but are processes that allow students to gain high level thinking and problem-solving abilities doubtlessly (Borasi, 1994). Errors are important components of education period considering the contemporary curriculums' purpose of raising individuals equipped with problem solving and high-level cognitive abilities. Teachers' exploitation of student error as a means of education, approach to error via intervention or conceptually is vital to fulfill such purpose. Putting student errors to use as such bolsters the learning process and bestows the ability to question the own learning status upon students (Borasi, 1989).

There are various uses of error approach in teaching. Errors can be used as a means of teaching (Heinze, 2005). Errors, which are depicted as springboards (Borasi, 1994), allow the identification of learning struggles of students, student' understanding teaching processes (Borasi, 1996) and establishment of cognitive conflict environment (Bell, Brekke & Swan, 1987). Exploitation of errors in teaching supports teachers' improving their own content knowledge (Akkusci, 2019; Borasi, 1986, 1989, 1994; Gedik & Konyalioglu, 2019; Ozkaya, 2015; Ozkaya & Konyalioglu, 2019) hence allow teachers to review (Kuntze & Reiss, 2006) their teaching knowledge. On the other hand, taking advantage of errors in teaching creates a positive impact on student success (Barbieri & Booth, 2020; Durkin & Rittle-Johnson, 2012; Heinze & Reiss, 2007; Rach et al., 2013; Rittle-Johnson & Star, 2009; Yildirim, 2019). Moreover, allows students not to refrain from making mistakes (Guzmán-Muñoz et al., 2009; Oser & Spychiger, 2005 as cited in Heinze & Reiss, 2007; Heinze, 2005) and transform the situation into an opportunity (Borasi, 1994; Ginat, 2003; Heinze, 2005; Heinze & Reiss, 2007). Therefore, the role of errors in education is indispensable. Nevertheless, teachers must identify their error approaches in order to use the concept of error in their teaching practice.

Approaching to errors conceptually necessitate predicting the type of errors student might make and knowing the root causes of these. The researchers conducted studies so as to identify error approaches due to the stated necessity and the impact of conceptual approach on education (Baker & Chick, 2006; Bilgili, Ozkaya, Ciltas & Konyalioglu, 2020; Crespo, 2000; Demirci et al., 2017). It has been seen that error approaches could not be determined and the reasons of identified errors were not explained correctly in some of these studies. Failure in error identification corresponds to a situation in which erroneous solutions and explanations are accepted as true. Additionally, it has been seen that the solutions were not evaluated properly with respect to their validity or falsity (Bilgili et al., 2020; Demirci et al., 2017). The conducted studies showed the evaluation of student errors is related to the understanding of student thoughts (Baker & Chick, 2006; Crespo, 2000). Knowing student thoughts is within the concept of understanding students (Ball et al., 2008; Hill, Ball & Schilling, 2008). In the recent years, thanks to the increase in interest to the knowledge of understanding students, the research that investigates identification of student errors and interventions to student errors became prominent (Son, 2013).

Despite the fact that error approach in terms of intervention is considered as a tool for teachers that enhances their teaching capacity, the number of studies on the matter is scarce (Chick & Baker, 2005; Son & Sinclair, 2010; Didis et al., 2016). Furthermore, the number of studies regarding the methods of in class interventions of teachers is limited as well. Leiß and Weigand (2005) propounded four distinct intervention methods regarding affective, metacognitive, related to the content and organization. Another parameter in the class environment which is governed by teacher interventions is error climate. Steuer et al. (2013) delineated a survey for the purpose of determining the error climate in classroom. It has been revealed that the personal reactions of students to errors are affected from the error climate in classroom. The identification of teacher intervention approaches to student errors became a necessity when the effect of teacher intervention and error climate in classroom are considered. Although limited, there are research in which teacher interventions to student errors were identified due to the necessity (Santagata, 2002; Santagata, 2004; Santagata, 2005; Turkdogan & Baki, 2012; Son, 2013; Didis Kabar & Amac, 2018). The previous studies showed that teacher candidates' intervention approaches to student errors were unsatisfactory (Didis et al., 2016). With respect to this, the interventions were comprised of pointing out of operative errors and their explanations (Son & Sinclair, 2010). Holmes, Miedema, Nieuwkoop and Haugen (2013) demonstrated that teachers opt for directly expressing errors of students instead of correcting their misunderstandings of concepts. There are examples of studies in which interventions such as directly expressing student error were identified. In these studies, approaches of teachers/candidate teachers to student errors were identified as asking questions, showing and indicating the true way, expressing the error, make students notice the error, not explaining, commenting, explaining the problem, re-teaching the subject, cognitive conflict and scrutinizing the student ideas (Chick & Baker, 2005; Didis et al, 2016; Didis Kabar & Amac, 2018; Son & Sinclair, 2010; Son, 2013; Turkdogan & Baki, 2012). Santagata (2002: 2004: 2005), who asserted that interventions to errors were shaped via educational practices and cultural beliefs, expressed similar approaches. For the purpose of identifying these approaches student solution manuals (Didis et al., 2016); teaching scenarios (Didis Kabar & Amac, 2018; Son & Sinclair, 2010; Son, 2013); in-class videos (Didis et al., 2016; Santagata, 2002; Santagata, 2004; Santagata, 2005; Turkdogan & Baki, 2012) and the students errors generated by the researchers (Chick & Baker, 2005) were employed. Recently, there has been an increase in video-based studies in which teachers are trained to improve teachers' approaches to student errors (Holmes et al., 2013; Santagata & Yeh, 2014).

2. 2. Teacher Noticing

Teacher noticing is accepted as a fundamental component of teaching expertise and student-centered education (Lampert, Beasley, Ghousseini, Kazemi & Franke, 2010; Schack, Fisher & Wilhelm, 2017; Sherin, Jacobs & Philipp, 2011; van Es, 2011). As a unique way of recognizing and interpreting the important aspects of classroom interactions, "teacher noticing" became the focal point of much research (e.g., Barnhart & van Es, 2015; Star & Strickland, 2008; Sun & van Es, 2015; Ozdemir Baki & Kilicoglu, 2020; van Es & Sherin, 2002). The related studies investigated the nature of noticing by

means of the theoretical facade of noticing and brought various designs that would support teachers' capability of noticing via the practical perspective of noticing as well (Dindyal, Schack, Choy & Sherin, 2021). The teacher noticing is a wide concept which attracted great interest and most of the studies on this concept focused on teachers who notice mathematical thoughts in the context of mathematical education (e.g., Jacobs, Lamb & Philipp, 2010; Walkoe, 2015). van Es and Sherin (2002) defined teacher noticing is the ability to identify in class interactions and associate linkages between noteworthy actions, whereas Rodgers (2002) defined it as analysing the learning progress of students and ability to provide feedback. Jacobs et. al. (2010) described the concept of noticing with three cascaded abilities pertinent to mathematics: (a) recognizing the strategy of the student, (b) evaluating the mathematical thinking of the student and (c) deciding on how to respond to the mathematical thinking of students. According to the stated definitions, it can be claimed that the ability of noticing is intimately related to the teaching activities.

The role of teachers in designing an effective teaching and its application is paramount. The responsibilities that teachers undertake is closely tied to their professional knowledge and skills (Dogan & Kılıc, 2019). In order to present an effective mathematics education teachers have to determine students' pre-knowledge, dominant understandings and misconceptions, interest and what type of supports they require (NCTM, 2000; Shulman, 1986). Henceforth, teachers can develop sensitive teaching applications by practicing on understanding students closely (Goldsmith & Seago, 2011; Jacobs et. al., 2010; Ozdemir Baki & Kilicoglu, 2020; Smith & Stein, 2011). Nonetheless, understanding the mathematical thinking students is a challenging task and requires more than identifying whether the responses of students are right or wrong (Callejo & Zapatera, 2017). Thus, comprehending and evaluating the mathematical thinking of students is closely related to an advanced capability of noticing. Respectively, one of the most effective methods of teachers to understand the student thoughts is error approach (Pirie & Kieren, 1994). A teacher can understand and analyse student error only if they adopt a correct way of intervention approach. This can only be achieved by deciding the proper intervention approach with embracing a questioning stance to student errors. Teachers' proper approach to student errors can be thought as a reflection of noticing abilities on teaching practices as well. With respect to this, Krupa, Huey, Lesseig, Casey and Monson (2017) claimed that only the teachers who notice errors and misunderstandings of student can establish the next step of teaching of students based on their thinking.

In the relevant literature there exists research concerning teachers' conceptual noticing skills of student errors (e.g., Aydın-Guc & Turker, 2021; Kılıc, 2019; Stephens, 2006). These studies indicated that teachers' or teacher candidates' abilities of associative thinking to student errors and misconceptions, interpretation and responding were inadequate. Apart from the analysis of student errors in the classroom, how teachers intervene to and tackle the errors must be emphasized (Santagata, 2002). Improving the quality of education can only be possible through teachers' being cognizant of student thinking and deciding on the appropriate intervention approach based on this noticing (Van Zoest, Stockero, Leatham, Peterson, Atanga & Ochieng, 2017). Therefore, in this study, teachers' approach to student errors and their noticing of these approaches were investigated by means of in class videos in concomitance of teacher noticing and intervention approach concepts.

3. Methodology

3. 1. Research Design

The study was conducted in cooperation with four middle school mathematics teachers who were employed in two different state schools located at east of Turkey in 2020-2021 academic calendar. The research was designed as a case study, which is a qualitative research method, to identify teachers' approach to student errors and their noticing of their approach. As such, the videos of lectures of the four middle school mathematics teachers were analysed so as to determine their approach to student errors. Moreover, the teachers' noticing of their approach to student errors on was investigated via online interviews (a semi structured interview concerning the behavior in the videos) with the teachers.

3. 2. Participants

The study was conducted with four middle school mathematics teachers who were employed at two state schools which are centrally located. The selection of the participant teachers was realized with criterion sampling which is one of the purposive sampling methods. Teachers' having at least three years of experience in the profession was taken as the criterion. Some studies in the related literature indicated that for the effectiveness of the education a three years of teaching experience is required (e.g., Erdik, 2014; Rivkin, Hanushek & Kain, 2005; Star & Strickland, 2008). On top of these, teachers' having at least three years of teaching experience is important for their recognition of students (student pre-knowledge and his misconceptions) as well. Both schools in which the teachers were employed were located in the same frame and the demographic properties of the students were similar. The schools' being centrally located and close to each other eased the process of video recording of the lectures. For the sake of ethical concerns, teachers were addressed as Bade, Meva, Fuat and Sevda which are not their real names. The demographic characteristics of the teachers are provided in Table 1.

Table 1. *Demographic characteristics of the participant teachers*

Teachers	Gender	Teaching Experience
Bade	Female	6
Meva	Female	11
Fuat	Male	10
Sevda	Female	9

3. 3. Data Collection

For the purpose of determining teachers' approach to student errors the lectures of the participant teachers were recorded three times at different dates. Therefore, one of the researchers attended the class with the teachers and recorded an hour (30 minutes due to the pandemic) mathematics lecture. The researcher endeavoured to identify the student errors and teachers' error approaches during in class interactions. The lectures of the teachers were recorded at least once in two weeks for three times. Subsequently, the researchers picked the situations in which there exist student errors and made shorter video sections. Thus, 16 video sections in which student errors and teachers' error approaches in terms of interventions exist were created. Of these 16 sections, teachers Bade has 6, Meva has 4, Fuat has 3 and Sevda has 3 sections. The length of the sections varies with respect to the teacher student interactions.

The most adopted approach by the teachers to student errors was identified by examining the sections. Afterwards, the teachers reviewed their video sections and their noticing of their own approach was questioned. On that purpose, online interviews were conducted with the teachers. In the interviews, first the researcher and the teacher watched the relevant section then the researcher posed the questions as follows: “Could you please explain your approach in the video you have watched to the student error?”, “Why did you decide on that approach?”, “Do you think that your approach was effective for student to understand his/her error?”, “Could you suggest another approach for the same situation?” and, “Why?”. Then, the teachers were asked to comment on the error approach of the other teachers whom they do not know in person via watching her/his video sections. Thereby, each teacher watched the same video section and was asked to comment on the situation and suggest alternative approaches.

The data collection tools of the study were the 16 video sections extracted from the lectures of the teachers and the interviews conducted with the participant teachers. The use of different data collections tools was important to examine the teachers' error approach in a holistic manner.

3. 4. Data Analysis

The data obtained in the study were analysed qualitatively (Creswell, 2018, p.182). Qualitative data analysis was realized as a descriptive data analysis. The video sections of the teachers were analysed in order to identify the error approach of the participant teachers. The researchers created a code list by exploiting the related literature (Chick & Baker, 2005; Didis et al., 2016; Ma, 2010; Son, 2013; Didis

Kabar & Amac, 2018; Son & Sinclair, 2010) in order to transform (Heritage, 1984) their qualitative data set to a structure on which they can work easier. The list is comprised of the following codes: "making the error noticed, making the truth noticed, asking questions, expressing/explaining the truth directly, expressing/explaining the error, not making an explanation, explaining the question, not intervening to the errors, re-explaining the subject". In the first phase of the study the two researchers coded based on the codes in the list independently and simultaneously. The consensus among the researchers was calculated as 91.4% (Miles & Huberman, 1994). There were differences detected in cases where more than one coding was realized. Subsequently, the perspective of the third researcher was sought in order to eliminate the identified differences. The coding structure was finalized by collecting the overlapping codes under the same category via the consensus of the three researchers. Thus, the validity of the coding was also demonstrated. The statements, which were created in the second phase of "reminding the related part" and "asking for peer opinion" were added to the code list whereas the codes "not intervening to the errors" and "re-explaining the subjects" were omitted from the list since they were not observed. The coding categories related to the observed adopted approaches of teachers to student errors were presented in Table 2.

Table 2. *The codes and categories regarding the teachers' intervention approach*

Categories	Codes	Teacher Behaviours
Questioning	Making the error noticed	Asking questions to students to make them cognizant of their error Making verification, making the student check the validity of the results Proposing counter examples regarding the error subject, creating contradiction
	Making the truth noticed	Dropping hints to indicate the true solution Asking questions to indicate the true solution Propounding a similar example and make the notice via the example
	Asking questions	Posing questions to ensure whether the subject was understood. Asking questions to unveil the student thinking. Asking questions to check the present students' knowledges
Description	Expressing/Explaining the truth directly	Expressing the concept, definition or operation belong to question.
	Expressing/explaining the error	Expressing their error directly to students
	Not making an explanation	Not making any explanation
	Explaining the question	Reading the question once more. Expressing what were given and asked in the question
Providing the information	Reminding the related part	Referring to the parts related to the question
Peer Support	Asking for peer opinion	Getting opinion from the peers

The one-to-one interviews with the participant teachers were analysed to identify the teacher noticing of their approach to student errors. The related literature (Jacobs et al., 2010; van Es & Sherin, 2002; van Es & Sherin, 2021) on teacher noticing were examined and tailored for the study. Therefore, the teachers' noticing of their error approach were investigated at two stages. These are the teacher's

evaluation of his/her error approach in relation to student error and evaluate his/her approach so as to determine what type of approach s/he can adopt.

In order to avoid the overlapping between the codes that were generated during the data analysis process and to keep the codes in an independent structure, the definitions were reviewed and the codes were evaluated once more. Moreover, the sentences transcribed from the video sections were chosen as the analysis unit. The validity of this research was ensured by collecting information via making the teachers watch their videos and conduct detailed queries with them, describing the data analysis process in detail and supporting the research outcomes and the statements in the interviews with direct citations (Creswell, 2018, p.254).

4. Results

In this section, the outcomes of the study which are the error approach of the teachers and their noticing on their adopted approach were represented in two sub chapters.

4. 1. The Error Approach of Teachers to Student Error

The teachers' approach to student errors were investigated under the categories of explaining, providing information, querying and peer support. 16 video sections in which an error situation present were extracted from the lectures of the teachers. The adopted approaches of the teachers were identified and their frequencies of applying for that approach is given in.

Table 3. *The participant teachers' error approach and the frequency of the adopted approaches student error*

Categories	**The Codes**	**The Participating Teachers**				
		Bade (f)	Meva (f)	Fuat (f)	Sevda (f)	**Total**
Questioning	Making the error noticed	9	5	6	9	**29**
	Making the truth noticed	15	4	2	3	**24**
	Asking questions	26	16	12	24	**78**
Description	Expressing/explaining the truth directly	1	3	-	2	**6**
	Expressing/explaining the error	4	5	-	2	**11**
	Not making an explanation	-	1	-	1	**2**
	Explaining the question	3	6	5	2	**16**
Providing the information	Reminding the related part	4	2	2	2	**10**
Peer Support	Asking for peer opinion	2	1	4	-	**7**
		64	**43**	**31**	**45**	**183**

The participant teachers opted for disparate approaches; nonetheless, the most preferred approach is asking questions which is in the questioning category (see Table 3). It was observed that the teachers opted for the asking question approach in order to determine whether the question was understood, to check current knowledge level of students and reveal the thinking of students. Furthermore, the teachers frequently opted for the approaches of asking questions to make students recognize their error, make students validate the result and propounding a counter argument so as to create a contradiction. Especially, three of the teachers (Meva, Fuat, Sevda) were inclined against the approach of making students recognize their error whereas Bade opted for asking questions in order to make students perceive the correct way and dropping hints to indicate the truth. In addition, it has been observed that the teachers employed more than one intervention type together. Concordantly, an exemplary dialogue regarding Bade's approach against a student's error during the activity of matching fractional expressions with percentages (see Figure 1) was given:

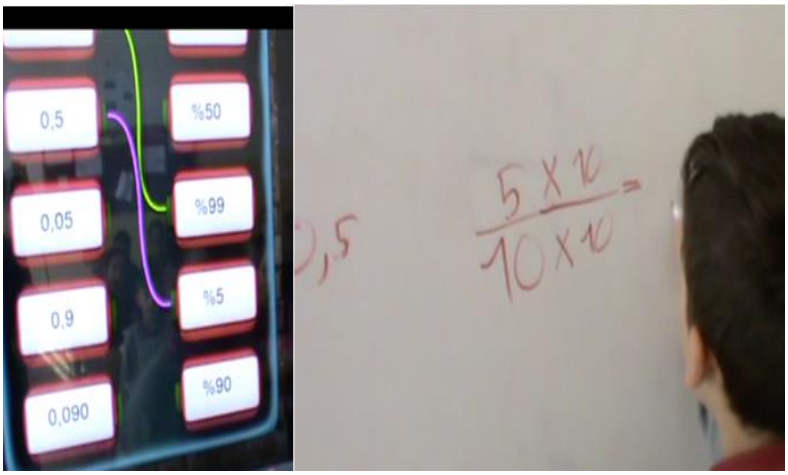


Figure 1. *A situation in which a student made an error during the matching activity*

The student matched 0.5 with 5%.

Bade: Hmmm. Let's see. Is 0.5 equal to 5%? Come on here Berat (student). Let us erase this side. Let's write 0.5.

Student: 5/10.

Bade: Let's write 0.5 on the front. Now, let's read the fraction.

Student: 5 divided by 10.

Bade: Now tell me what is the denominator of the fraction?

Student: 10.

Bade: Alright. Is it possible we can express it as a percentage since its denominator is 10?

Student: (wonders a little) We can only show the numerator with the percentage symbol.

Bade: But how are you going make 10 out of numerator? What can you do?

Student: What if I expand it?

Bade: Yes. I mean, can we directly write 0.5 and can we call it 5%? Let us see where we made an error.

Student: $\frac{5 \times 10}{10 \times 10} = \frac{50}{100}$.

Bade: Look what you have marked and what result you get. You've marked 5% before, now what did you find as the result?

Student: I've found 50%.

As it can be seen in the dialogue Bade attempted to make the student recognize his/her error by querying whether 0.5 is equal to 5%. Therefore, she asked the student to read the fraction s/he wrote. The student reported that the denominator of the fraction s/he wrote was 10. The teacher posed the question *"...if it is 10 can we express it as a percentage?"* in order to make the student recognize his/her error. In response to the student's feedback Bade posed questions *"How are you going to do? What can you do?"* to check the knowledge set of the student. Moreover, the teacher attempted to indicate the correct way to the student by her approach as well. Subsequent to the student's providing the correct answer, the teacher *made the expression "Let us see where did we make an error"* to draw attention to the error and *posed the question "What have you marked ... but what did you find?"* to aid the student understanding the correct way.

In another video section of Bade, a student matched 0.9 with the expression 9%. But, after a short while of reflection, the student retracted and marked 90% instead of 9%. Teacher posed the question *"Why did you change?"* to unveil the thinking process of the student. The student written down the fractional expression then reached 90% via expansion. Concordantly, Bade reminded subject on fractional expressions then resumed the lecture.

In the category of description, the approaches of explaining the truth directly, explaining the error, not making an explanation and explaining on the question took place (see Table 3). The most preferred intervention approach under this category was explaining the question. The teachers had the tendency of reading the question one more time and emphasizing on the given and asked values in the question to student errors. Especially Meva and Fuat adopted this strategy via either by emphasizing the given and asked values themselves or requesting students to tell the values in the question. On the other hand, Meva was inclined to tell the error in certain error situations whereas she opted for direct explanation in other cases. Accordingly, an exemplary dialogue which demonstrates (see Figure 2). Meva's approach against a student who made an error while trying to calculate the area of perpendicular trapezoid with another method.

Student: What if we draw the height from D to B?

Meva: There is no height from there. Eren (the student) tells that if we connect D and B (drew the diagonal of the perpendicular trapezoid). He says can we find the area of DBC triangle. Let us see how he is going to find it. Now I draw this and here is 14 cm. What will you find there? You will find two areas, right? Then let us call these areas as A1 and A2. Now, find me A1 area.

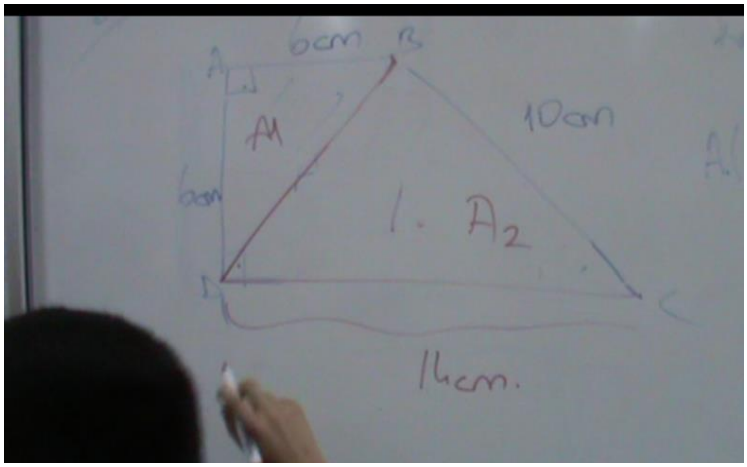


Figure 2. *A situation in which Meva shows her explanation in writing*

In response to the erroneous thinking of student, Meva expressed the error of student by saying that there cannot be height between two points. Moreover, she did not elaborate on why there cannot be a height between the points. Starting from the student's idea of connecting two corners she asked how he was going to find the resulting areas in order to control the student's current knowledge set. She comprehended the student's confusion from his gestures then indicated the given values in the question again and adopted the approach of explaining the question via naming the areas created by the drawn diagonal. The teacher's interaction with the student continued as below:

Student: A1 area... (held the pencil but they could not proceed.)

Meva: A1 is a perpendicular triangle; correct? As we did previously; what were we calling the perpendicular sides of the triangle as?

Student: (No response.)

Meva: Were they not called height as well? If our base is 6.

Student: (Drew the height belonged to hypotenuse.)

Meva: Hold on, it is not needed. Now, the base of this is it (indicated); write it.

Student: (wrote 6.)

Meva: Now we have something perpendicular on this base. What is its relation to the base?

Student: Its height.

Meva: Yes, it is.

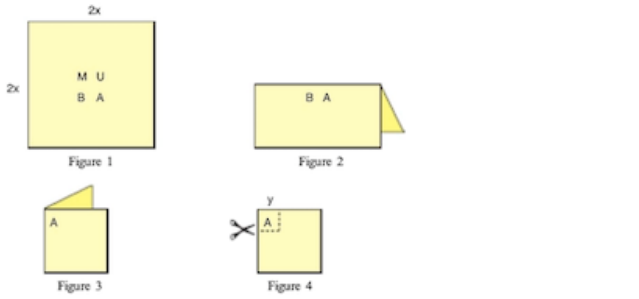
Student: $6 \times 6 = 36$

Meva: That is correct but what have we missed every time? (Took the pencil from the student and added a division by 2.)

Student: $36/2 = 18$.

As it can be seen from the dialogue, although the student propounded an alternative method to solve the question, his/her erroneous thinking on the height concept prevented him/her from solving the question correctly. Despite the fact that Meva tried hint that the perpendicular sides are also the height, the student tried to draw a height onto the hypotenuse. The questions posed by the teacher oriented the student to the truth instead of revealing the thinking process of the student. The student multiplied the lengths, which are 6 cm of the two perpendicular sides in order to find the area of the triangle but made an error by not dividing the result by two. The teacher emphasized this situation by the expression *"we have missed"* and showed the truth by writing it.

Another approach adopted by the teachers to the student errors is reminding the related part. The teachers adopted this strategy in cases in which students stalled while solving the question or after they have finished solving. For instance, a student in Bade lecture multiplied 55 and 2.1 and found the result as 1155. The teacher posed the question "*So, is the result of this 1155?"* in order to make the student notice the error. After some thought process, the student gave the answer 11.55. With the question *"Will you put the comma there?"* the teacher kept her approach of making the student recognize the error. Then student realized the error and refined the answer as 115.5. This once, the teacher inquired so as to reveal the thinking process of the student and expressed as "*We were conducting multiplication as if there was no comma then were placing the comma with respect to the total number of digits after the comma."* to remind the related part. Similarly, Fuat drew the student's attention to the error then following finishing the solving reminded the related part. Furthermore, at all three video sections he called for the peer support to make the student recognize the error instead of directly telling the truth. For example, he shared the question given in Figure 3 about equality and asked them to solve it.



A square paper is being folded from top to bottom as in Figure 2, and then in half from left to right as in Figure 3, with the corners overlapping. The square piece on which the letter A is written, shown with dashed lines in Figure 4 on the folded paper, is cut and discarded and the paper is opened.

The algebraic expression that gives the area of one side of the remaining paper is equal to which of the following?

A) (2x-y)(2x-y) B) (2x-y)(2x+y) C) (2x-2y)(2x+2y) D) (x-2y)(x+2y)

Figure 3. *The equality question* (Edited from the 8th grade mathematics textbook of MUBA publications)

Fuat: Who wants to come to solve the question?

Student: Teacher, the multiplication of these is x^2. Let me explain this, there will be a gap like this. If we call here y^2, we will have $4y^2$ since we have four parts. Since the question asks for the remaining area, we have to subtract its area. So, its area $4x^2 - 4y^2$ which results in $2x^2 - 2y^2$, hence the answer is C.

Fuat: Now, you have written $4x^2 - 4y^2$ equals to $2x^2 - 2y^2$. Well, do you think that these two expressions (pointed the expressions) are equal to each other?

Student: No teacher. There shouldn't be any squares. We get $2x - 2y$.

Fuat: How did you get there? Let us rewrite the expression of $4x^2 - 4y^2$ and see how can you write an equality with it.

Student: Teacher, x^2 expressions cancel out and we get $2x - 2y$.

Fuat: Why the squares cancel out? Why did you think that?

Student: Because x cannot be written.

Fuat: But there is no division here. I want to know what is $4x^2 - 4y^2$ equals to. So, you came up with $2x - 2y$ but how did you get there?

Student: Its equivalent is this (option C).

Fuat: Alright, how will you show that option C is its equivalent? I mean, why not option A, B or D but you chose C?

Student: Teacher, it says $4x^2$ here. Then (points out option C) it has to be $2x$ and $2x$. It says $4y^2$ here as well. So, it has to be $2y$ and $2y$ here as well.

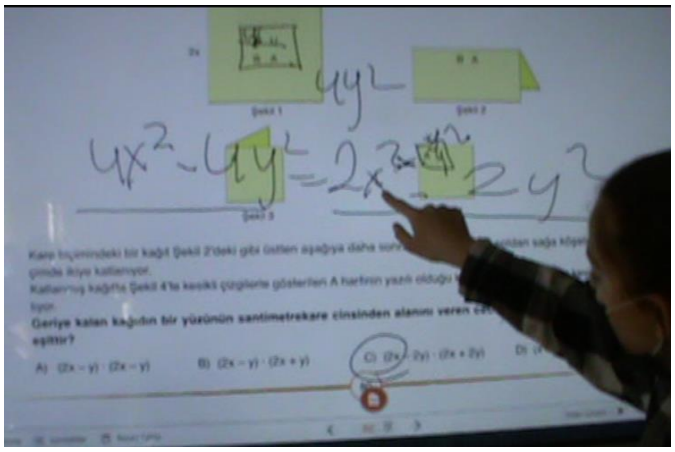


Figure 4. *A situation in which a student made an error while writing the equality of difference of two squares*

As seen in Figure 4, the student made an error by writing the difference of two squares as $4x^2 - 4y^2 = 2x^2 - 2y^2$. In response to the error, Fuat posed the questions *"Are these two expressions equal?", "How did you get there", "Why would the squares cancel out?"* in order to make the student recognize his/her error. The student reached $2x - 2y$ as a result by referring to vanishing of x^2 against the teacher's questions. Then, Fuat understood that the student was referring to division operation and stressed out that there is no division. Therefore, the teacher opted for the approach of asking question to reveal the thinking process of the student. The student was persistent on the option C. Fuat called for peer support to the student's erroneous thinking even though he was pointing out the correct option. Then the dialogues evolved as follows:

Fuat: You think like that. Alright fellows, who wants to verify that this expression equals to option C? Yes, Kevser (student).

Student: In order to expand the difference of two squares, (pointing out $4x^2 - 4y^2 = 2x - 2y.2x + 2y$)

Fuat: It was the identity of two squares. Alright, how do we ensure that they are equal the difference of two squares.

Student: The product of these is equal to that.

Fuat: Well, how did you get there? For example, I am asking this. Why didn't you write $4x^2 - 4y^2 = (4x - 4y)(4x + 4y)$ but came up with $2x - 2y.2x + 2y$?

The student who responded made an error while writing the difference of two squares by not putting parentheses. Fuat posed questions to reveal how she reached that conclusion. By propounding a counter argument to make the student recognize her error. Meanwhile, the teacher also called for peer opinions. Subsequent to a few students' expressing their thoughts, a student emphasized that the use of parenthesis was important and made the student recognize her error. Then Fuat adopted the approach of reminding

the related part by saying "*... when we take the square of* $2x$ *we are taking both the squares of 2 and x together, however, when we say x square, only the square of x...*"

The teachers' approach to student errors were not only limited to operational errors but also against erroneous thinking. For instance, Sevda asked a student who solved the question of an arithmetic mean correctly what that result means and unveiled the erroneous thinking of the student related to the concept of arithmetic mean. The dialogue between the teacher and student evolved as follows:

Sevda: Alright, what is arithmetic mean? Do you remember?

Student: Yes. We take some numbers, do some stuff and turn them against us.

Sevda: Could you explain what does that mean?

Student: Hmm. Let me give my examination scores as example. From the examinations I took 85, 95 and 100. I add these up then divide by three and the result is 93.

Sevda: Alright, what does the result imply? What does 93 correspond to?

Student: This number indicates that my grades got better.

Sevda: Alright, what if you had 100, 95 and 85 from the examinations respectively, what would be your grade average?

Student: (Reaches 93 by mathematical operations.) Would be 93 again.

Sevda: Hmm. Does this indicate that your grades got better?

Student: No, it did not show that my grades got better, it got worse.

Sevda: Although your grades decreased your grade average became 93. It was 93 again when you grades got better as well. Then, what does 93 demonstrated to us?

Student: That my grades remained the same.

Sevda posed questions in order to reveal the thinking process of the students. The student explained the arithmetic mean as the increase of her grades by being deceived by the sequence of her grades. Then the teacher requested to calculate the arithmetic mean for the counter example she gave. Consequently, the student found the same result once more but stated that her grades plummeted. Therefore, she reached conclusion that arithmetic mean is independent from the sequence of the grades. In the pursuing dialogues Sevda ensured that students learnt meaningfully. She asked the students to comment on the arithmetic mean of the counter example she created.

4. 2. The Teachers' Noticing to Their Approach

For the purpose of demonstrating the teachers' noticing of their approach against student, short video sections from their lectures were shown. The integrity among the outcomes were strived to be established by displaying the video sections of the most adopted approach of the teachers. The teachers' noticing on their error approach were analyzed in the frame of evaluating student errors and offering an alternative approach for the same situation.

The participant teachers commented on their error approach in the relevant video sections. The teachers' tendency was to define and explain their error approach. Furthermore, they made critical and deductive comments on their approaches. For example, Bade saw the student's error whilst matching the fractional expression with percentage expression and in order to make the student notice his/her error s/he made interventions as observed in the video sections. Accordingly, the teacher did not tell the error immediately, instead, s/he posed further questions so as to make the student aware of the error, moreover, these questions contained hints towards the truth. The teacher remarked that her approach was effective in the recognition of the error and commented as follows: *"... I adopted an appropriate approach* "Then she made further deductions on her approach. Below are some excerpts from Bade's commentary:

> *"The student corrected the error. But I guided a little. First, I posed questions to reveal the error then to make him/her see his/her error. I think it was effective since s/he corrected the mistake on his/her own."*

Bade thinks that her approach of asking questions was effective in unveiling the error of the student and make the student correct it. She expressed that with her approach she drew attention to the error whilst keeping in mind that there has to be others who did the same error in the class and it would be enlightening for them too. Therefore, she said that she will employ the approach of asking questions in order to make the error conspicuous and hint the correct way.

Meva said that she gave away the truth without allowing the student who proposed an alternative method for calculation of trapezoid to recognize the error. Later she noticed that she asked questions to make the student see the height belongs to hypotenuse, yet, she answered them without receiving a reply from the student. Moreover, she remarked that student made an error by not dividing by two whilst calculating the area of triangle and she showed the correct answer herself. Meva's commentary on her approach to student errors are as follows:

> *"The kid must have not understood the concept of height since he thought that it is about connecting two corners. He proposed a method without knowing how... I guided the student more than enough, hence, he couldn't understand what he did and how. I immediately intervened and divide by two instead of him. Honestly, I did not give the opportunity to recognize his error. I assisted, yet I directly gave the answer since he couldn't see on his own."*

She expressed that her approach was not informative and she will employ another approach to the same errors. She expressed that first, she will ask student to draw the height with explaining the height concept. Subsequently, she will explain the part regarding height subject once more and elaborate on the relation between area calculation of triangle and rectangle. Hence, it has been seen that Meva will adopt the approach of re-expressing the subject as an alternative approach to student errors.

Fuat expressed that the thinking of the student was erroneous and he asked questions to reveal how this error was made. He indicated that his objective was to allow the student to recognize his error and reasoned his approach as below:

> *"... the student equated the expression of $4x^2$- $4y^2$ to $2x^2$- $2y^2$ Then he chose the right option by expressing that it is difference of two squares. However, I was curious about how he made the connection. Hence, I asked questions to reveal his thinking process. I asked the student to tell me where he did the error and called for peer support for further discussion."*

Then he further discussed the effectiveness of his approach. He remarked that he finds his approach effective since he was able to make the student recognize his mistake and ensured in class participation. Regarding this he expressed as follows:

> *"The kid voluntarily admitted that he made an error and he is confident that he can learn the truth, hence, I totally believe that my approach was effective. I believe the approach was constructive since they reached the solution on his own together with his friends."*

Fuat's comments are that it is effective to use questioning and peer support approaches in order to realize the student's erroneous thinking. He observed that the student opted for memorization instead of understanding the subject background. He propounded that the best way to reveal thinking process is to ask questions. Therefore, Fuat's commentary on his approach is dependent on the content of student error. Moreover, he expressed that he will also employ the approach of re-expressing the subject while making further suggestions regarding his video sections. On the issue of identifying if there are similar errors in class "*...I would pose a tricky question to reveal if there is trouble. I would ask students to comment on the question together and re-express the subject then ask further question to strengthen their understanding."*

Sevda teacher indicated that due her noticing of misconception ("We take some numbers, do some stuff and turn them towards us") error of the student even if the student did not make an operational error posed questions to reveal thinking process of the student. She remarked that instead of expressing the error of the student she endeavored to understand erroneous thinking of the student. Moreover, she tried reveal what the student understood from arithmetic mean. Sevda teacher remarked that the approach she adopted was effective to the student error. She elaborated on the validity of her approach by giving the example of the student's gained ability to interpret the results on her own based on her better understanding of arithmetic mean. Later on, she asked teacher to find the arithmetic mean of another

number set and interpret the result. However, she noticed that the student was having trouble in interpreting the result.

> *This time I determined the grade set as 100, 95 and 5. I was curious about the student's interpretation of the result. The student calculated the result as 66 and was surprised about it. Because the solution she found in the previous question was close to the grades. But this one was not. Therefore, she had trouble in interpreting.*

Sevda adopted the questioning approach against the misconception of student on arithmetic mean and corrected the student error as such. Hence, she finds her approach effective. Nonetheless, she noticed that her giving contradictive examples without elaborating on the same subject caused confusion. Therefore, she decided that giving complementary examples and then going for counter examples will be a better approach.

Subsequent to the teachers' watching video sections of their own lectures they watched a short section of lecture of another teacher whom they do not know. In the video, the student claimed $\frac{19.6}{0.0014}$ he has to remove the comma to do the operation and in order to do that he has to multiply with 1000. With respect to this, teacher asked questions to help student to find the correct way. Fuat expressed that the teacher in the video section made too many interventions. He reflected his thought as *"... he did too many guiding statements so that the student did the operations without understanding." I don't think the teacher recognized the reason of the error..."* He claimed that the approach of the teacher was not effective in making the student recognize the error since the student did not understand the rationale behind the operations he did. Fuat implied that he would adopt another approach against the same error. Firstly, he expressed that he would wait until the end of division operation to understand how the student performs. Secondly, he said that he would pose questions to remind the prior knowledge in order to make the student recognize his error. Moreover, he remarked that he would call for peer support by discussing the error with class. Sevda reckoned that the student could not establish the connection between number of digits and zeros after comma. She mentioned that the teacher was too goal oriented and his questions for reaching to the result instead of making the student understand the error. Regarding this she quoted as follows: *"... the student said that he was going to multiply with 1000. When the teacher objected with increasing his voice tone while he was asking about the multiplication, the student erred again and opt for 10000."* In addition, she further remarked that the teachers goal oriented guidance even though his being aware of the situation was not an appropriate approach. So much so that, she mentioned that this approach would cause confusion among other students as well. Hence, she concluded that it was not an educator approach and began to explain her preferred approach for the case. She suggested that she would inquire prerequisite knowledge of the student with fundamental questions since the student could not figure out multiplication due his deficient understanding in comma expressions. She quoted that she would pose questions such as "*Why do we eliminate comma? How would you perform this division?"* to understand the thinking process of the student. She said that she would go for complementary examples that would help the student to establish the connection between powers of ten and number of digits after comma. Bade opinion was that the teacher guided the student to the correct result with his questions. She claimed that the teacher did not allow the student to recognize his error and explaining the truth directly. Henceforth, she did not find the teacher's approach apt and suggested the approach of questioning. Accordingly, she remarked that she would not intervene the student's multiplication. Moreover, she said that she would request the student to perform division by writing fraction expressions as rational as an alternative. Hence, she indicated that she could measure the student's knowledge set. Lastly, Meva's opinion was similar, and she deemed that the teacher did not allow the student to correct his error. She expressed that her preferred approach would be to re-expressing the subject then ask the student to solve the question again. Hence, it can be inferred that Meva teacher would opt for a teacher centered approach than a student-centered approach.

5. Discussion

The research was conducted in order to identify teachers' approach to student errors and their noticing on their approaches. As such, the videos of lectures of the four middle school mathematics teachers were analyzed so as to determine their approach to student errors. One of the findings of this study is that the

teachers possess more than one approach with respect to error. Likewise, Chick and Baker (2005) and Didis et al. (2016) demonstrated that teachers and candidate teachers have multiple approaches to student errors. It has been observed that the teacher (Bade) adopted the highest number of approaches has the least experience. This can be interpreted as the number of approaches used are not in a correlation with occupational experience.

The teachers' approach to student errors were investigated under the categories of questioning, description, providing the information and peer support. It is one of the key findings of the study. Approach of teachers/candidate teachers were subjected different classifications in many studies (e.g, Chick & Baker, 2005; Didis et al., 2016; Didis Kabar & Amac, 2018). In this study, the categories and the error approaches in the categories (the codes) are similar, yet another construct was created depending on the teachers' response to student errors.

The most popular error approach among the teachers was within the category of questioning. In this study, they tend to be questioning to student errors. Furthermore, it was observed that the questions they posed to questions in order to make a questioning were not satisfactory. Questionings are important so as to reveal mathematical thinking of students (Crespo & Nicol, 2003). In this context, it can be claimed that the teachers could not completely evaluate student thinking. This result is similar to the study of Didis et al. (2016).

The most adopted approaches among the questioning category within the teachers were asking questions, making the error noticed and making the truth noticed. The teachers determined whether students understood the questions by asking and revealed the student's pre-knowledge on the subject and their thinking processes. This can interpret as it is hard for teachers to understand student errors during class. Didis Kabar and Amac (2018) have reached a similar conclusion with written student scenarios. Another important finding of the study is that all the teachers except Bade, adopted the approach of making the error noticed. It has been identified that these teachers opt for different strategies (cognitive conflict, check solution, ask question to make the error noticed) while indicating the error. Similar strategies were seen in the studies of Turkdogan and Baki (2012) and Santagata (2002). Besides making the error noticed within the category of questioning is making the truth noticed which was adopted by Bade. It has been seen that Bade mostly employs strategies of dropping hints and giving similar examples (Figure 1 and relevant dialogue). In the relevant literature it has been shown that the aforementioned strategies were employed by candidate teachers/teachers in order to make the truth noticed (Santagata, 2002; Turkdogan & Baki, 2012; Didis et al., 2016).

The most adopted approaches among the category of description among the teachers are explaining the question, expressing the error, expressing the truth directly and not making an explanation respectively. Whilst the teachers explain questions to student, they either reiterated the given and asked values or repeated the question. The teachers' interventions of the category of description became telling the truth or error directly where explaining the question is insufficient. This is an important finding of the study. In the literature, it has been revealed that candidate teachers use this approach mostly (Son, 2013; Son & Sinclair, 2010). The reason for this was propounded as candidate teachers' who employ the approaches of description do not know how to query student errors and their incapability in proposing educative questions (Didis et. al., 2016). The teachers in study opted for such approaches only for certain student errors. This indicates that teachers need more in-depth knowledge of the subject where the student made an error. The last preferred approaches (reminding the related part, asking for peer opinion) are within the category of providing the information and peer support. In the light of the findings, the least preferred approach is call for peer opinion. Perhaps, it was not adopted by teachers since its application is challenging.

It has been seen that the teachers provided descriptive and explanatory commentaries on their error approaches when their noticing on their approaches were investigated. Moreover, they made inferences about the efficacy of the chosen approach to student errors. In the relevant literature there exists findings on the effect of error approaches on making student more courage in making errors (e.g., Guzmán-Muñoz et al., 2009; Oser & Spychiger, 2005 as cited in Heinze & Reiss, 2007). Three of the teachers (Bade, Fuat ve Sevda) were content of their error approaches whereas Meva admitted that she did not pick an appropriate approach. This points out that the teachers have the ability to evaluate their own

error approaches objectively. Teachers who can make these assessments can improve their approaches via productive self-criticism and take decisions on their teaching methods (Krupa et al., 2017). In this study, it has been seen that the teachers who noticed student errors and made productive comments adopted student centered approaches.

Another important outcome of the study is that the teachers' suggestion of explaining the subject to student errors which indicates the teachers have the tendency of taking their occupational decisions as teacher centered. The previous studies on noticing student errors expressed that the teachers are incompetent in understanding student errors and misconceptions, hence, in responding to them (e.g, Kılıc, 2019; Stephens, 2006). Therefore, in order to improve teachers' noticing ability, more actions and incentives are needed (Sherin, Russ & Colestock, 2011). Teachers' noticing ability of student errors reflects on in class activities (Ozdemir Baki & Kilicoglu, 2020) and enables teachers to decide on proper error approaches (Van Zoest et al., 2017). As such, the probability of teachers' understanding student errors is much lower if teachers could not recognize certain details (Him Lam & Kam Ho Chan, 2020).

All the participant teachers made adverse commentaries on error approaches of the teacher they do not know in person. Therefore, it was inferred that the teachers are in consensus on not giving the truth immediately without allowing students reflect on their errors. Moreover, the commentaries of the teachers comprised of pieces of student thinking and on inferential level. This is in coherence with other studies on teachers' making comprehensive remarks on other teachers' lectures (e.g., Seidel, Stürmer, Blomberg, Kobarg & Schwindt, 2011). It is significant that the suggestive approaches of the teachers are analogues to their adopted approaches. Erickson (2011) propounded that the variety of teachers' commentaries is related to their occupational experience and pedagogical approach.

6. Conclusion

In conclusion, error approaches of middle school mathematics teachers were classified within the frame of this study. It has been seen that the teachers are making questioning on both student errors and student misconceptions. On the other hand, the study was not focused on a sole subject. Therefore, it is significant that the teachers opt for the approaches in description category (e.g., explaining the truth or error directly) in some certain matters (Figure 2 and the dialogue). For this reason, it is thought that teachers' pedagogical competencies can be examined in the context of the subject. The finding of this study will be especially important since it was conducted in cooperation with experienced teachers unlike the majority of other studies. Furthermore, the teachers' watching the video sections of their own lectures and making commentaries on their approaches was an important experience for them. As such, successor studies can envisage video-based strategies towards improving teachers' approaches to student errors by means of taking this study as a model.

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Authors

Gülşah Özdemir Baki, Oltu Faculty of Humanities and Social Sciences, Ataturk University, Erzurum, Turkey, E-mail: gulsah.baki@atauni.edu.tr

Merve Özkaya, Department of mathematics education, Kazım Karabekir Faculty of Education, Atatürk University, Erzurum, Turkey, E-mail: mdurkaya@atauni.edu.tr

Alper Cihan Konyalıoğlu, Department of mathematics education, Kazım Karabekir Faculty of Education, Atatürk University, Erzurum, Turkey, E-mail: ackonyali@atauni.edu.tr