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# Representations in Primary Mathematics Teaching

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**Abstract:** The OECD PISA (Programme for International Student Assessment) investigates whether students have acquired the applicable knowledge essential for full participation in a modern society (they measure how students can apply their knowledge to novel situations). Meaningful learning and understanding are basic aspects of all kinds of learning and they are even more important in the case of learning mathematics. The aim of this research was to measure students' independent thinking and problem-solving skills, as well as to investigate the way they can actively apply their knowledge when solving problems directly and not directly connected to the curriculum. We have investigated the relationship between different knowledge areas and levels in the case of Primary School and Kindergarten Teacher Training College's students at Partium Christian University Oradea.

**Key words:** teacher training, mathematics education, arithmetical problem-solving methods, primary school textbooks

## Introduction

International assessments in science and mathematics carried out during the 1970s and 1980s were based around the curriculum and investigated how students aquire disciplinary knowledge and how they apply it in a context similar to those encountered in the classroom. In the '90s The OECD encompassing the most developed countries in the world launched the three-yearly PISA (Programme for International Student Assessment) in order to investigate whether students possess applicable knowlegde in three knowledge areas (reading, mathematics and science) essential for a full participation in a modern society. [16]

The results of the assessment always indicate the areas displaying negative trends; they give rise to different analyses. The final conclusion is always the fact that mathematics needs to be promoted more actively, reading should be given a greater focus, and since the teacher plays an important role in students' performance, teachers should be trained and selected accordingly. Specialists say that well-trained teachers are the most efficient in developing students' individual skills. PISA 2012 mathematics score rank Asian countries or regions (Shanghai, Singapore, Hong Kong, Taiwan, Korea, Macao, Japan) on the first seven places. In the '80s the American researcher, Harold Stevenson investigated the reason for better results in mathematics in Asian countries as compared to Americans. He came to the conclusion that it is due to the well-trained, dedicated teachers using good methods for teaching mathematics. Learning used to have a prestige, knowledge opened the way to social advancement. Mathematics was particularly suitable for a breakthrough: not even unfavorable social background could pose a hindrance in the way of talent. [7]

## Reading

Reading, meaningful learning, acquisition, and understanding are the basic aspects of all kinds of learning. They are perhaps even more important in the case of learning mathematics.

In our teacher training programme we have placed great emphasis on word problems, on their correct interpretation, understanding, on observing the steps in problem solving, possible representations, interpreting results in terms of real world situations, etc. [11, 12]

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Research shows that teachers rely on coursebooks most frequently when teaching (63.7% of respondents), accompanied by workbooks (76. 9%), exercise books, study guide (37.3%), programme (32.5%), and other (9%). [3] According to a research carried out in 2011 on teachers of primary schools in Satu Mare county, they were most content with the only Hungarian coursebook for 4th grade. They considered it the best due to its scientific accuracy, language, problems, utility, coherence and for providing facilitated self evaluation [2]. This book turned out to contain the most versatile problems in an appropriate number. The exercises are differentiated, the coursebook contains an appropriate number of logical problems, and the problems are realistic.

During the methodology seminars students solved most of the problems from the book, and we analyzed the possible incorrect definitions (especially the ones in geometry).

The book contains altogether 818 problems, 76 logical problems (9.29%) (at the bottom of every second page). 636 (77.75%) problems require text interpretation and only 182 (22.25%) problems require operations, these were practice problems.

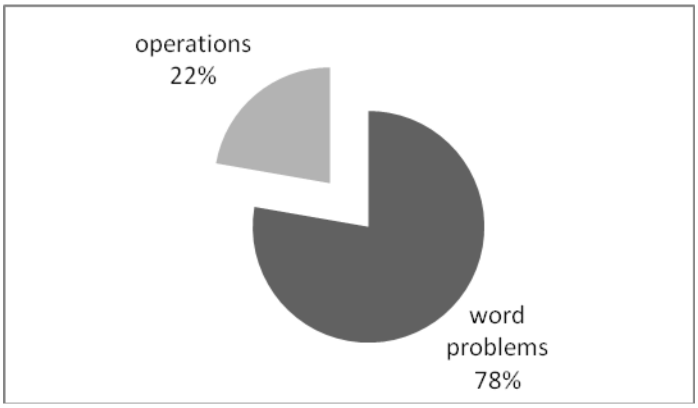


**Figure 1**. Distribution of the problems in the coursebook for 4th grade [10]

## Representations

Most of the information we receive about the world around us is filtered through our eyes. The visual plays an important role in our life. Hence, visual representations play an important role in the learning process. Numerous psychological studies confirm that using visuals in teaching helps a deeper understanding of concepts. People tend to remember the visual aspects of a concept better than analytical aspects.

“For a mathematical thinking and communication we need to repesent in some way the elements of mathematical structures. Communication requires external representation in the form of language resources, written symbols, figures and objects”. [8] External representations can be: enactive, iconic and symbolic (written and spoken language, symbols).

In order to conceptualize about a mathematical concept we need its internal (mental) representation, so that our brain can operate with these representations. As opposed to external representation internal representation cannot be directly observed. Cognitive psychologists have formulated two hypotheses on representations:

1) There is a connection between internal and external representation of a concept. We can make logical deduction about internal representation, about their quality with the help of manipulating external representations.

2) Internal representations are interconnected, they form a network, that of mathematical concepts and principals. These connections can be simulated by constructing the right connections between external representations [1]. External representations, such as figures and text definitions influence the nature of

internal representation. This also holds the other way round. The way a student reperesents his/her knowlegde externaly shows the way he/she represents the information internaly [5]

Symbolic representation is the most compact and abstract representation of a principle or concept. On the other hand, enactive and iconic representation provide a better understanding of the essence and importance of a concept or principle, they facilitate sense making. Visual representation may help understanding. People remember the visual aspects of a concept better than its analytical aspects because memory operates better with images than words [1]. Teaching the three types of external representations in a spiral movement would be optimal. The learning process is affected in a positive way if relying on different cognitive styles, integrating verbal, analytical and visual activities. Zoltán Dienes's multiple embodiment principle posits that in order to understand abstract concepts a multiple representation and manipulation of these representations is needed [4]. Visual representation often facilitates understanding a problem. Students need to be taught a conscious use of visual representations. Those who excel at problem solving choose the best representation for a certain problem. They easily use a geometric representation for a problem in algebra [1].

"Using concrete and iconic representations is necessary not only for the so called slow students or elementary students. These representations are important for all students and are useful throughout the entire learning process". [15]

Traditional didactics states that iconic and enactive representations are important in the early stages of learning, and as students' age increases symbolic representation should take over. However, the view that iconic representation should be implemented at all stages is gaining more and more ground [1].

"One mode of representation does not suffice for the conditions and requirements of solving a problem or managing a situation. Most often multiple representations is asked for. A parallel engagement of different modes of representation and the connection between these yields a more efficient activity. Mathematical power lies in the properties separate from representations and the connection between representations". [6] Pictorial representation is a particular method in primary education arithmetics which can be used when solving word problems. The essence lies in representing the data of the problem, the unknown and the relations between them, and using the representation to analyze and solve the problem. One can use sketches, plane figures, segments, symbols and conventional signs or letters. Representation is important since it contributes to a better understanding and memorization of the problem. [13]

In our teacher training programme we place great emphasis on different methods of solving arithmetic problems, especially on particular methods (such as representation, contrasting, hypotheses, backwards working, rule of three, method of balancing) and drills. Teacher trainees need to be thought to see through a child's eyes and they need to part with the algebraic methods as they need to adapt to children's way of thinking, and should not use unknowns represented by letters of the latin alphabet.

First of all they need to be retaught the arithmetical methods and we need to promote these as opposed to algebraic methods.

Practice shows that this is not an easy endevour. Students favour the algebraic methods, which they are more familiar with, and have a hard time with representations although, they need representations for advancing better understanding, and mathematical reasoning. Most of the word problems in textbooks require representation and an even larger number require a special problem- solving method.

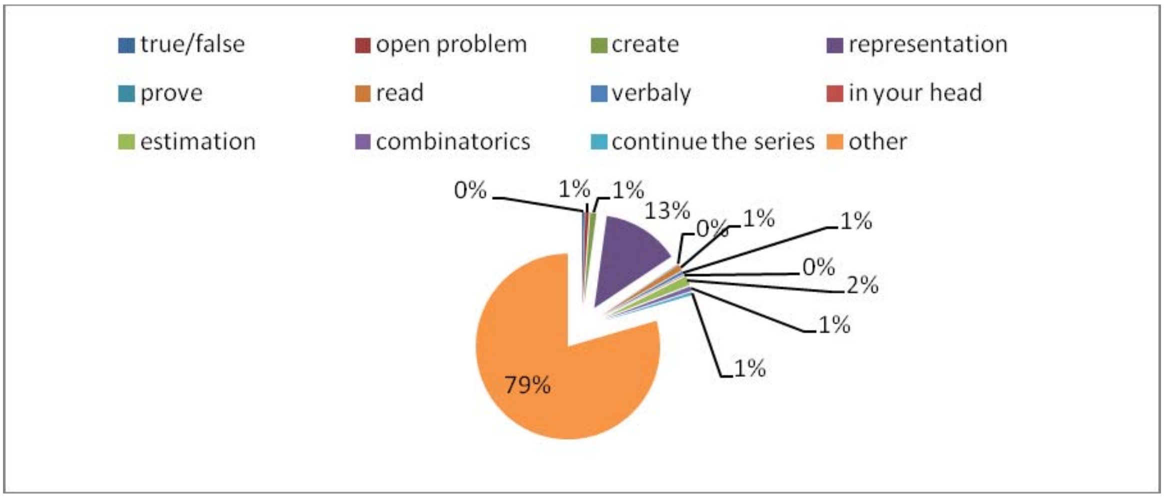


**Figure 2.** Division of problem types in the 4th grade coursebook [10]

When analyzing the various problems in the above mentiones coursebook [10] we found the following types: problems requiring representation: nearly 100 (12.22%), estimation 12 (1.61%), reading 8 (1%), create 8 (1%), combinatorics 6 (0.8%), open problem 5(0.67%), verbaly 4 (0.53%), continue the series 4 (0.53%), true/false 3 (0.4%), in your head 1 (0.13%) , prove 1 (0.13%), other 590 (79.51%). (15.64%- of the problems required entry, while 84.35% did not)

## The research

**Aim**. By asking students to solve problems related to the curriculum our aim was to measure their individual thinking, problem-solving skills and how they can actively apply their knowledge, after a semester of training. In the case of various word problems we asked for arithmetic solution. We wanted to asses the extent to wich they have acquired this method following their course on mathematical methodology. We would have liked them not to choose the algebraic solution, since this method is not applicable to the age group (7-11 years, grade 1-4) they are goint to teach. We hypothesized that we had managed to convince students only partialy to use representations and arithmetic methods. We wanted to investigate the connection between different knowledge areas, levels (operations, conceptual understanding, problem and exercise solving), hypothesizing a causal relationship.

**Methodology**. We asked our kindergarten and primary teacher trainees (23 second year students) to complete a test in mathematics. The problems in the test are appropriate for testing usable knowledge, since they require careful reading and understanding.

The test contains 11 problems, i.e. 11 items in six groups. By means of these problems we investigated students' knowledge of the basic mathematical concepts and their operational background, as well as students' interpretation of word problems and their use of representations, i.e. the application of mathematics in everyday life.

The test is similar to the ones used in schools for assessing content knowledge. Problems are on a basic level (practically part of the 1st-8th grade curriculum), however they test the skills component of knowledge. Apart from six items, the problems do not directly ask for curriculum knowledge. Students need to understand, interpret and make connection between elements in order to solve the problems. These have to be solidly integrated into their knowledge system. The test is primarily suitable for displaying simple, quantitative information and analyzing problems of understanding. In the case of word problems requiring representation it was also an important aspect whether students can solve them using various representations (as they are expected when they become teachers) or whether they can only apply the algebraic method.

When selecting the problems it was an important aspect that they should not be rich in mathematical content. We chose problems which are essential due to their applicability in other subjects or fields.

Students needed to use their knowledge of: the concept of fractions, fraction of a whole number, operations with fractions, probability calculation, combinatorics, permutation, measurements, area calculation, arithmetic mean, logical value, etc.)

Based on their content the problems form six groups. Each group focuses on different problems of understanding; nonetheless each problem requires text interpretation.

Word problems that can be solved using arithmetic methods (representation with segments) (1, 6): These are the most basic readings, operations, without which one cannot make even the simplest calculations. In order to understand word problems and use a correct representation students need to be familiar with some common terms, how much more, how many times more, as well as to read connections correctly (somewhat more, less, mathematical operations: addition, subtraction, division, multiplication).

Problem solving (2, 8): combinatorics- counting all possible situations, probability counting, comparing the probability of two cases (which is more probable?)

Operations (4, 9): In these problems students have to deal with the concept of fractions, fraction of a whole number, fraction of a fraction, as well as operations with fractions.

Logical problems (5, 11): These exercises are slightly more difficult, they require a more complex reasoning. Students need to understand the concept of the arithmetic mean, and have to assign logical values. Problem 11 is a logical problem.

Word problems (3, 7): A legs and heads problem (students could use symbolic representation), and a word problem requiring operation with fractions.

Geometry problem (10): Students need to be familiar with the concept of the perimeter of a rectangle. Representation can be used.

Students were given one hour to complete the test. Answers, i.e. each item was rated on a dichotomous scale (right/wrong). Students scored 1 point for right answers and 0 points for wrong answers.

## The test used for assessment

1. Bence knows that a pen costs 1 zed more than a pencil. His friend bought 2 pens and 3 pencils for 17 zeds. How many zeds does Bence need if he wants to buy 1 pen and 2 pencils? [16]

Arithmetic solution:

The pencil costs: I---------I

thus, the pen costs: I---------I----I.

two pens and three pencils cost: 17 { I---------I---------I----I----I

I---------I---------I---------I.

17-2=15, 15:5=3.

As a result, a pencil costs 3 zeds, while a pen costs one zed more, i.e. 4 zeds.

2. A smaller box contains 20 tickets, numbered 1-20. A larger box contains 100 tickets, numbered 1-100. Without looking draw a ticket from a box. From which box are you more likely to draw ticket no. 17? [16]

A) The box containing 20 tickets.

B) The box containing 100 tickets.

C) The probability is the same for both boxes.

D) One cannot tell.

Right solution: A) because $p_1 = \frac{1}{20} > p_2 = \frac{1}{100}$ .

3. A concert ticket costs 10, 15 or 30 zeds. 900 tickets have been sold. $\frac{1}{5}$ cost 30 zeds, $\frac{2}{3}$ cost 15 zeds. What fraction of the tickets sold costs 10 zeds? [16]

Solution: $1 - \left(\frac{1}{5} + \frac{2}{3}\right) - 1 - \frac{13}{15} - \frac{2}{15}$.

4. Diana bakes a blueberry cake which is one and a half time larger than the amount in the original recipe. If according to the original recipe we need $\frac{3}{4}$ cup of sugar. How many cups of sugar does Diana's cake need? [16]

Solution: $\frac{3}{4} \cdot \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}$.

5. A car salesman has posted an advertisment in the newspaper: "Old and new cars for sale, various prices, 5000 zed avarage price. " Based on the advertisment which of the following can be true? [16]

A) The price of most cars is between 4000 and 6000 zed.

B) Half of the cars cost less than 5000 zed, while half of them cost more than 5000 zed.

C) Az least one car costs 5000 zed.

D) Some cars cost less than 5000 zed.

6. Three crates contain altogether 614 kg of goods. The second crate is twice as heavy as the first one and 4 kg lighter than the third one. How many kg of goods do the crates contain? [10]

Arithmetic solution:

First crate is 1.: I---------I

thus, the second is: 2.: I---------I---------I

while the third: 3. : I---------I---------I----I 4 kg heavier.

Altogether 614 kg of goods.

614-4=610. 610:5=122.

As a result the first crate weighs 122 kg, the second 122x2=244, while the third 4 kg more, i.e. 244+4=248 kg.

7. There are altogether 30 poultry and sheep in my grandparents' yard. Knowing that there is a total of 70 legs how many sheep and poultry does my grandmother have? [10]

Solution: symbolic representation is required. Each animal has at least two legs, thus a total of 30x2=60 legs. The difference: 70-60=10 sheep legs. Thus, there are 10:2=5 sheep and 30-5=25 poultry.

8. Anna, Béla, Csilla and Dóra are going to the cinema together. In how many different combinations can they seat on four chairs next to each other? Write down the possible seating orders. [10]

Solution: 4x3x2x1=24 combinations..

9. Five children share four oranges.How much does one child get? [10]

Solution: $\frac{4}{5}$ orange for a child.

10. A rectangle-shaped, 42 cm long and 27 cm wide picture is glued on a cardboard which is 5 cm larger than the picture in all directions. Calculate the perimeter of the picture and that of the cardboard. [10]

Solution: the dimensions of the picture w=27, l=42, perimeter P=2l+2w= 2x42+2x27=138 cm.

The dimensions of the cardboard w=27+5=32, l=42+5=47, perimeter P=2l+2w= 2x47+2x32=158 cm.

11. Logical problem: There are 10 pieces of socks drying in the yard. We want to pick up a pair of socks in the dark. We know that there are five different pairs of socks. How many pieces do we have to pick up to make sure that we pick up a pair? [10]

## The results of the survey

The table below contains the test results. The problems were chosen in conformity with the curriculum, they were solvable, thus results can be compared to the highest possible score, the one hundred per cent achievement.

**Table 1.** Avarage score and dispersion of the test

| Number of students | Avarage score | Dispersion |
|---|---|---|
| 23 | 5.60 (50.98%) | 2.74 |

The extreme values are the following: one student (4.3%) scored zero points(0%), another student (4.34%) performed much better than the rest, scoring a maximum of 11 points, i.e 100%. Two students (8.69%) achieved more than 90% on the test. (Considering that the test contained simple problems the results are mediocre).

**Table 2.** Results of the sections and the test (scores and percentage, total)

| Sections | Maximum score | Number of right answers | Section results in percentage |
|---|---|---|---|
| 1.Word problems requiring arithmetic methods | 46 | 25 | 54.34% |
| 2. Problem solving | 46 | 35 | 76.08% |
| 3.Operations | 46 | 22 | 47.82% |
| 4. Logical exercises | 46 | 14 | 30.43% |
| 5. Word problems | 46 | 22 | 47.82% |
| 6.Geometry problem | 23 | 11 | 47.82% |
| Test (total) | 253 | 129 | 50.98% |

The first section contained word problems requiring arithmetic method (representation with segments) (1, 6). We asked for the arithmetic solution because we wanted to test to what extent students managed to aquire this method following their one semester course in mathematical methodology. We wanted to avoid the algebraic solution, since this method cannot be used for the age group they are going to teach (7-11 years, 1st-4th grade)

Students solved problem 1 in the following way:

**Table 3.** Methods chosen by students for solving problem 1

| Method | Arithmetic | Algebraic | Guessing, trials | Wrong or no answer |
|---|---|---|---|---|
| Proportion of students who applied this method | 7 students (30.43%) | 5 students (21. 73%) | 7 students (30. 43%) | 4 students (17. 39%) |

.

In the case of problem 6, 13 students (56.52%) solved the problem correctly using arithmetic method, 4 students (17.39%) used correct representation with segments (accurate reading) but failed to solve the problem, while 6 students (26.08%) did neither use representation nor solve the problem.

In the problem solving section students did not seem to have problems with probability calculation. 95.65% of students solved the problem, only one student (4.34%) did not. In the case of the combinatorics problem, where students had to calculate all possible combinations and write down the possible seating orders, 13 students (56.52%) made accurate calculations, while 10 (43.47%) did not, and failed to write down the correct seating orders. Practically this was a permutation calculation, however when teaching elementary students we need to understand the logic of counting, i.e. to write down all the possible seating orders, setting forth the shortened counting methods.

Operations (4,9): Problem 4 asked for calculating the fraction of a fraction. 10 students (43.47%) gave correct solutions, while 13 (56.52%) failed, even though the problem presented an everyday application.

Problem 9, five children sharing four oranges, asks for a deeper understanding of fractions. 12 students (52.17%) made a correct division (8 students (34.78%) gave the solution in the form of common fraction, while 4 students (17.39%) , provided a decimal fraction). 11 students (47.82%) failed to solve the problem correctly. It has been pointed out in class that decimal fractions are not part of the elementary curriculum, thus students should not have used them.

Logical problems require a more complex reasoning. In problem 5 students had to assign logical values. 18 students (78.26%) failed to assign logical values correctly (they have problems with the arithmetic mean). Only 5 students (21.73%) solved the problem correctly. Problem 11 was a logical problem. 14 students (60.86%) failed to solve it. 39.13% solved it correctly. All things considered, students have difficulties with logical problems. 69.56% did not know how to handle it. Only 30.43% were successful.

Problem 3 asked for operations with fractions. 13 students (56.52%) were successful, 10 students

(43.47%) failed. The heads and legs problem (Problem 7) was used to teach a problem solving method (it was not obligatory, but students could use symbolic representation). 9 students (39.13%) used symbolic representation, 14 students (60.86%) did not. Some students tried guessing (3 students-13.04%) but without results.

The geometry problem (Problem 10) asked for calculating the perimeter of two rectangulars. In order to do this the dimensions of one of the rectangulars had to be defined. 11 students (47.82%) succeeded in calculating the perimeteres, while 12 students (52.17%) failed.

Table 4 shows the correlation between sections. The Pearson correlation coefficients are very dissimilar in strenght. The strongest connection is between solving word problems and the knowledge of arithmetic methods (r = 0.64). Solving word problems is the most closely related to operations (r = 0.43). There is also a strong connection between operations and the use of arithmetics methods (r = 0.53).

**Table 4.** Correlations of the sections

| Section | 1. Arithmetic method | 2. Problem solving | 3. Operations | 4. Logical problems | 5. Word problems | 6. Geometry problem |
|---|---|---|---|---|---|---|
| **1. Arithmetic method** | ------------- | 0.30 | 0.53 | 0.30 | 0.64 | 0.25 |
| **2. Problem solving** | | ----------- | 0.04 | 0.07 | 0.30 | 0.04 |
| **3. Operations** | | | ----------- | 0.40 | 0.43 | 0.06 |
| **4. Logical problems** | | | | ----------- | 0.10 | 0.26 |
| **5. Word problems** | | | | | ----------- | 0.25 |
| **6. Geometry problem** | | | | | | ------------ |

## Conclusions

All things considered the test could be solved relying on elementary mathematical knowledge. It contained simple problems, the results are mediocre.

Analyzing the connections between the different sections of the comprehension test it becomes evident that students performed differently on the various sections.

Overall, 52.16% of the students solved the word problems correctly, while 47. 82% did not solve it (they struggled with comprehension and did not manage to learn the methods or learned them only partially). Those who solved it used representation (30.43%) or the algebraic method (21.73%). 30.43% of the students used representation with segments. 30.73% managed to solve it in such a way that would be ideal for their future teaching of mathematics. It is almost impossible to change students' fixed, inflexible mathematical reasoning in the course of one semester.

Students achieved mediocre results in operations (47.82%), geometrical application (47.82%) solving word problems (47.82%), and combinatorics 56.52%. The best results achieved in probability calculation 95.65%. Logical problems gave rise to the poorest results. Only 30.43% of the students managed to solve the problem. Students are prejudiced when it comes to these problems, they feel intimidated.

As a result of our investigation we found a causal relationship bewteen different knowlegde areas. The

strongest connection is between solving word problems and the knowledge of arithmetic methods (r = 0.64). Solving word problems is the most closely related to operations (r = 0.43). If students have acquired arithmetic methods there is more chance for them, as well as for children, for successful problem solving, since these are more basic, concrete methods facilitating understanding. Solving problems also requires an accurate knowlege of operations. There is also a strong connection between operations and the use of arithmetic methods (r = 0.53). This also emphasises the importance of problem solving.

If we want to become efficient we need all methods, tools and representations which help teachers to make mathematics lessons understandable and accessible for all students. Experience comes sometimes from individual work, sometimes from cooperation, students' problem solving skills are boosted and they become more active in class.

If children were taught accurate reading and different problem solving (arithmetic) methods which are adapted for their needs and their level of knowledge, they would become more successful in solving problems and they would have more positive and less negative experiences when it comes to learning mathematics.

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