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VIEWING THE ROOTS OF POLYNOMIAL FUNCTIONS IN COMPLEX VARIABLE: THE USE OF GEOGEBRA AND THE CAS MAPLE

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Abstract: Admittedly, the Fundamental Theorem of Calculus – TFA holds an important role in the Complex Analysis - CA, as well as in other mathematical branches. In this article, we bring a discussion about the TFA, the Rouché´s theorem and the winding number with the intention to analyze the roots of a polynomial equation. We propose also a description for a method that involves a complementary perspective for the use of two mathematical softwares. Thus, with the intention of cheking the TFA and other mathematical properties related with the polynomial equation roots, we enphazise the visualization, an exploration with the Geogebra and the computer algebric system - CAS Maple. This CAS permits execute complex algebric operations and the visualization of tridimensional objects. On the other hand, the software Geogebra makes possible the dynamic mathematical exploration of the sciencific concepts in the teaching context of CA.

Key words: Fundamental Theorem of Algebra, Visualization, Software Geogebra, CAS Maple, Winding number.

1. Introduction

In academic locus, we find several scientifics topics that can promote some barriers to the teaching and mathematical learning (Alves, 2012). In particular, we highlight in this paper some topics in the Complex Analysis (Atiyah, 2002, p. 2). Indeed, we observe few papers about the teaching of Complex Analysis. In Brazil, we constate some works about the teaching/learning of CA. With a great attention, we realize the abstract character of several concepts in this particular mathematical branch. Moreover, there are inumerous classical theorems and concepts that require a deep conceptual knowledged.

Nowadays, we can not deny the possibilities and advances in technology. Thus, in a specic way, when we turn to the teaching in Mathematics, we constate softwares that allows to exploitation of certains abstract contents (Alves, 2013a, 2013b). In this article, we take into account a complementary character of to use the system Geogebra and the CAS Maple in the academic teaching context.

Originating in these two arguments of the previous paragraphs, we put in evidence the perspective advocated by Needham (2000). He defends, describes and suggests an approach for to CA supported by the visualization and our perception about the graphic-geometric properties. In fact, in your book intituled "*Visual Complex Analysis*", we constate an approach that promotes a heuristic interpretation for the: power series of fonctions; the notion of complex derivative $f'(z)$; theory of integration; behavior of the graphics of the function in the complex variable, theory of residue, etc..

Here, we express a particular interest in a classical theorem that provides implications in various branches of Mathematics. Indeed, the Fundamental Theorem of Algebra – TFA allows to conclude a strong propertie about the corp of complex numbers□ . On the other hand, we cannot discuss it without mentioning the other two elements. The first is just the theorem that implies that one of possible statements and the proof for the TFA (Bottazzini, 1986). We talk about the Rouché´s theorem. The second element is a relevant topological argument called for "Argument Principle".

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We will start to discuss some properties and barriers related to the FTA. But, we are mentioning other fundamental notion in the CA. In fact, Needham (2000, p. 338) coments that “as the name suggests, the winding number $\nu[L,O]$ of a closed loop L about the origin O is simply the net number of revolutions of the direction of z as it traces out L once in its given sense.” This author seeks to transmit his ideas not only by the formalism and the prove process related to the theorems.

With a heuristic intention, Needham (2000, p. 338) shows the figures below. He describes a visual and perceptual procedure with the purpose to charaterize and determines the number associated with the symbol $\nu[L,O] \in \square$. In this sense, we detache his words when explain that “you can verify this values by starting at a random point on each curve and tracing it out with your finger; starting with zero, add one after positive (=counterclockwise) revolution of the vector connecting the origin to your finger, and subtract one after each negative (=clockwise) revolution.”. Certainly, these indications are not the simplest, because they involve complex mathematical ideas. See the figures below.

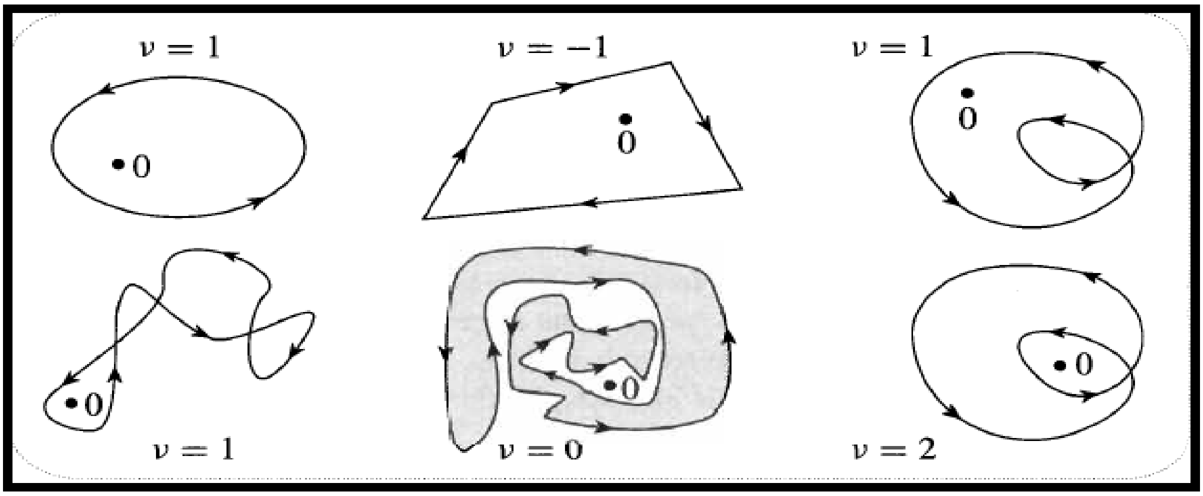

Figure 1. *Needham (2000, p. 338-339) proposes a heuristic procedural for to determine the winding number*

Well, we can not disconsider the topological ideas envolved with the winding number. Indeed, the notion of “simple loop” can actually be very complicated. Related to this fact, Needham (2000, p. 339) adds that “it seems clear, thought it is hard to prove, that it will divide the plane into just two sets, its inside and its outside. However, in the case of a loop that is not simple, such as [2], it is no longer obvious which points are to be considered.” (see figure 2-I)

On the other hand, given the topological complexity of the notion of “simple loop”, “the winding number allows us make the desired distinction clearly” (Needham, 2000, p. 239). So, we show two figures discuted by Nedham in your book. In the figure 2-I, we see an exemple of the loop that is not simple (on the left side). We visualize a typical loop wich determines a partition of the $\square - plane$ relatively to the sets D_1, D_2, D_3, D_4 (see [2]-I).

In the figure 2-II, we visualize a figure that transmits a visual meaning for to find the “winding numbers quickly”. About it, Needham (2000, p. 340) pointed some barriers that may hinder this process. He futher explains that “in [2] we found the winding number directly from the definition; we strenuously followed the curve with our finger (or eye) and counted revolutions. For a really complicated loop this could literally become a headache.” Needham develops a procedure for to perform the windin number. However, we observe in the static figure (2-II) that his method can become complicated. This is a problem that we can overcome with the aid of technology!

Specifically, in the next section, we will indicate certains limitations of the Dynamic System Geogebra which can be overcome with the Maple´s´ help. On the other hand, the static character related to the graphs provided by this CAS can be replaced by the dynamic aspect of Geogebra (Alves, 2012).

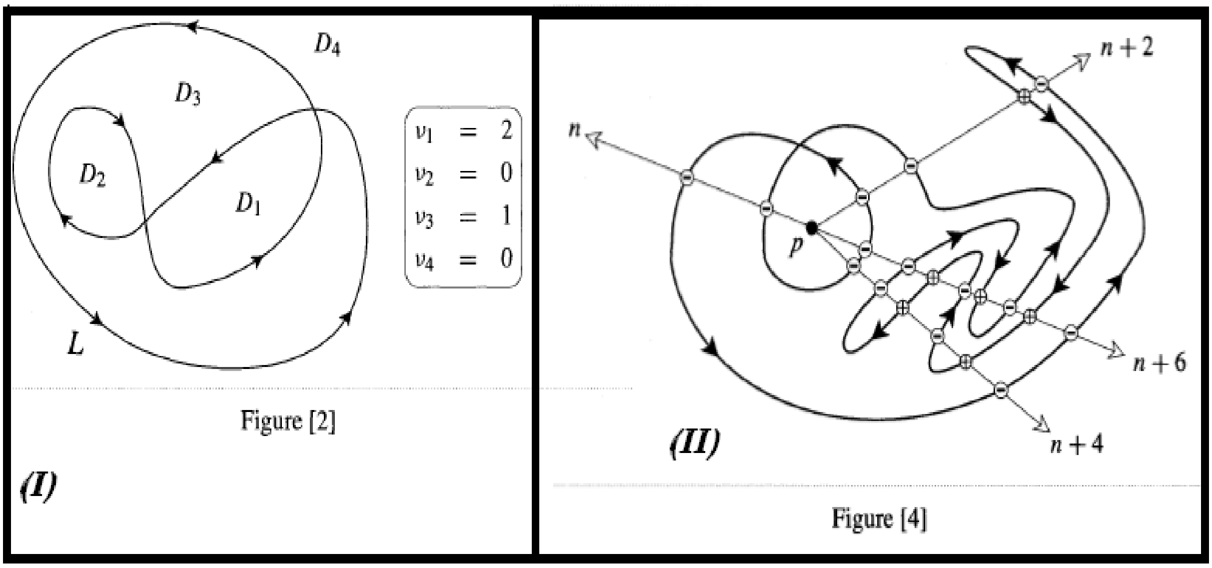

Figure 2. *Needham (2000, p. 341) proposes a method based on counting the number on intersections from a ray emanating of a point 'p'*

After this short discusion, the reader can ask where the TFA was. In the specialized literature, we realize a strong relationship beween the TFA and the winding number (Bottazzini, 1986; Needham, 2000; Kleiner, 2007; Krantz, 2007; Shakarchi, 2000; Stillwell, 1989; Tauvel, 2006). In particular, we will discuss the TFA in the context of analysis of the polynomial fonctions and the Argument Principle - AP. Thus, with an intention of describing a technique to perform the the study of the polynomial equation, we will indicate a way for the utilization to software Geogebra and the CAS Maple. From the mathematical point of view, we need to discuss too the Rouché´ theorem, but also the notion of winding number $\nu[L,O] = n \in \mathbb{Z}$. (Krantz, 2007, p. 139)

In the next section, we will notice the mathematical elements involved in the description of the use the two mentioned softwares. However, we emphasize the dynamical exploration of the topological elements provided by the software Geogebra. At the end of this paper, we propose a roadmap of actions that can be replicated by a mathematical teacher at the university, without much knowledge of a specific syntax of programs, however with a great concern on teaching and learning mathematics.

2. The Visual Complex Analysis and the study of the roots of a polynomial fonction

Influenced by Needham (2000), we employed in this work the terminology "Visual Complex Analysis - CVA". In their view, we observe several elements that put in evidence the visual-geometric properties. In fact, when we consider the transition process of the real to complex variable, we see radical changes about the meaning and the description of certains mathematical objects and process. However, before developing different arguments, we will stablish the TFA.

Theorem 1: The field $\mathbb{C}$ of complex numbers if algebraically closed. (Menini & Oystaeyen, 2004, p. 463).

Krantz (2007, p. 98) provides the polynomial $p(z) = (z-5)^5(z+2)^8(z-7)(z+6)$. In this case is immediate identification of its roots. In other cases, when we not have a factorization like this, from the mathematical point of view; we employ some analytical or a numerical method. Needham (2000, p. 56) presents an approach that emphasizes the graphic-geometric interpretation of the complex variable fonctions, indicated by $f(z) = f(x+iy) = \mathrm{Re}(f(x+iy) + i\,\mathrm{Im}(f(x+iy))$.

In the traditionnaly way, one of the methods to analyse the graph of a real variable function $f(x)$ consist in the study of the curve in the plane, for all values $x \in Dom(f(x))$. The elements are described in the set $(x, f(x)) \in IR^2$. However, when we turn to study of $f(z)$, with the condition

$z = x + iy \in \mathbb{C}$, we have to understanding the behavior of the following set $(z, f(z)) = (x, y, R(f(z)), \mathrm{Im}(f(z)) = (x, y, u(x, y), v(x, y)) \subset IR^4$. About this radical changes, Needham (2000, p. 56) explains that "note that although two-dimensional space is needed to draw the graph of a real function f, the graph itself is only one-dimensional curve, meaning that only one real number is needed to identify each point within it. Likewise, although four-dimensional space is needed to draw the set of the points with coordinates $(x, y, u, v) = (z, f(z))$, the graph itself is two-dimensional."

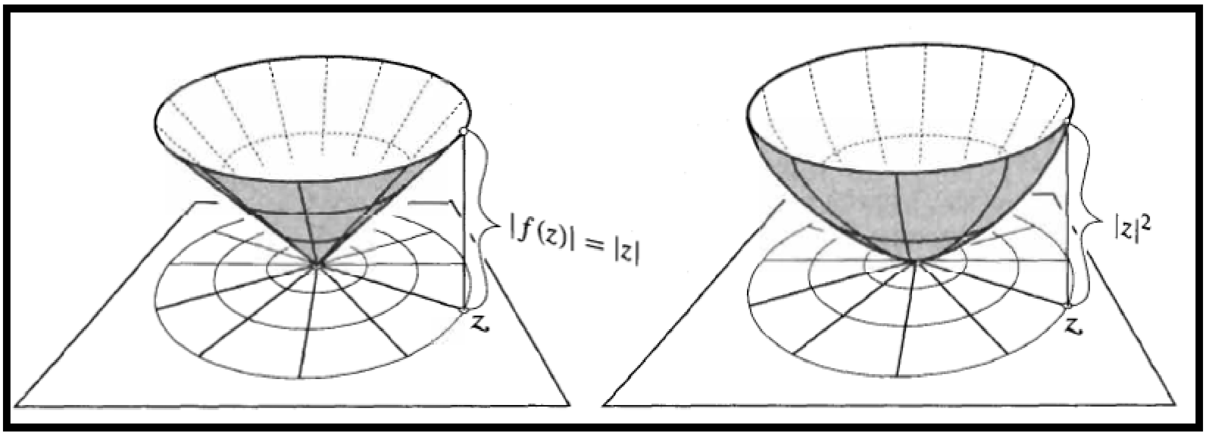

Figure 3. *The visual method proposed by Needham (2000, p. 56) for to describe graphically a complex variable function* $f(z)$ *by the modular function* $|f(z)|$

There are many techniques for visualization of the complex variable fonctions. Here we will use a technique wich involves the determination of the region in the plane or a neighborhood where we hope to find the roots of equations like $|p(z)| = 0$. For exemple, Needham (2000, p. 61) explains that "the Cassinian curves arise naturally in the context of complex polynomials". In fact, a general quadratic $Q(z) = z^2 + pz + q = (z - a_1)(z - a_2)$. For some particular cases, we can visualize, with the CAS Maple, the graphic-geométric behavior related to real and imaginary parts. In the figure 4, we observe some surfaces and differentiate the surfaces indicated by $\mathrm{Re}(Q(z))$, $\mathrm{Im}(Q(z))$, $|\mathrm{Re}(Q(z))|$, $|\mathrm{Im}(Q(z))|$. We included also the view from above, corresponding to the behavior of level curves (in color blue). As shown in Needham (2000, p. 56-57), its easly to identify the location of the roots when we take a modular fonctions. In the figure 4, we have considered $Q(z) = (z - 1 - i)(z + 2 + 3i)$.

We will use yet another mathematical concept that help us to identify the neighborhoods in the plane where we expect to find the polynomial equation roots. In this sense, when we turn to specialized literature, we already know that the level curves of the harmonic fonctions describe orthogonal trajectories. Thus, from the analysis of the behavior of the intersection of the level curves we can hypothesize the location of the roots of a polynomial function $f(z)$, with $z \in \mathbb{C}$.

Needham (2000, p. 354) explain that "the Argument Principle then inform us that: se $|g(z)| < |f(z)|$ on Γ, then $(f + g)$ must have the same number of zeros inside Γ as f.". From this characterization, we can prove the TFA. Indeed, lets consider $p(z) = z^n + Az^{n-1} + Bz^{n-2} + \cdots + E$ $(z \in \mathbb{C})$. Needham (2000, p. 354) employs the Rouché´s theorem for demonstrate the TFA. He took the polynomials $f(z) = z^n$ and $g(z) = Az^{n-1} + \cdots + E$, where $p(z) = z^n + Az^{n-1} + Bz^{n-2} + \cdots + E = f(z) + g(z)$. We must choose a circle Γ defined by $|z| = 1 + |A| + |B| + \cdots + |E|$. Finally, Needham (2000, p. 354) concludes "since f has n roots inside Γ (all at the origin), Rouché says that $p(z)$ must too.

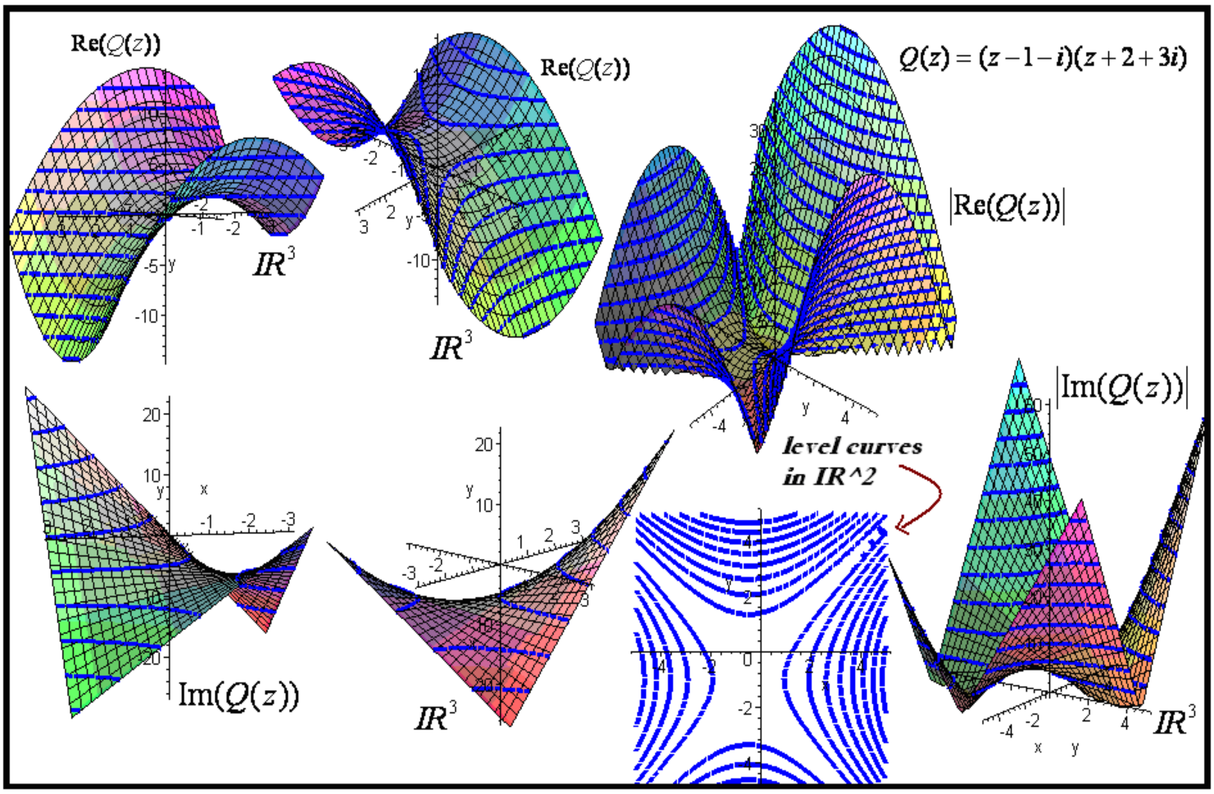

Figure 4. We visualyse the graphic-geometric behavior of the real and imaginary parts in IR^3

We emphasized this theorem that, traditionally, is discussed in the formalist style in almost all brasilian universities. However; we will indicate, with the techonologie´s help, another mode of exploring such theorem from the viewpoint of visualization and qualitative properties.

In the figure 5, we see a construction with the Geogebra that enables the identification and exploration of some geometrical and topological elements. In fact, Needham (2000, p. 341-342) discusses the behavior of a fixed loop and a continuously movind point. He observes that "the winding number only changes when the point crosses the loop". On the other hand, we can consider a fixed point and a continuously moving loop. In this case, "the winding number of the evolving loop only change if it crosses the point, ansd changes by ± 1 according to the same crossing rule as before". We can verify this dynamical and topological property in the figure below!

Well, we consider in the figure 5, the function $f(z) = z(z^2+1)(z^2-1)(z^2+4)(z^2-4)$. In virtue the TFA, we expect to find nine roots. With some basic commands of the Geogebra´s software, we describe a loop which intersects itself. On the right side, in the figure 5, we also indicate the orientation for the loop and, thus, we can count the numbers of revolutions around the origin O and others points in the plane. On the left side, we seek to adjust a region in the $\square - plane$ which is expected to find all or at least part of the polynomial equation roots $f(z) = 0$.

From the software capabilites, we can easily describe a continuously moving point as well as a continuously moving loop. We will show in the next section a way to promote the understanding about several topological properties that we can verify with the intention to confirm the TFA. So, in the figure 5, we observe the moving points P_1 and P_2. When we verify the winding number on the actually position, we count that $\nu[f(z), P_1] = 3$. On the other hand, when we put this point in other position P_2, this number can varies. With this purpose, we will show how exlore the capabilities of Geogebra and investigate the relationship between the roots and this topological notions.

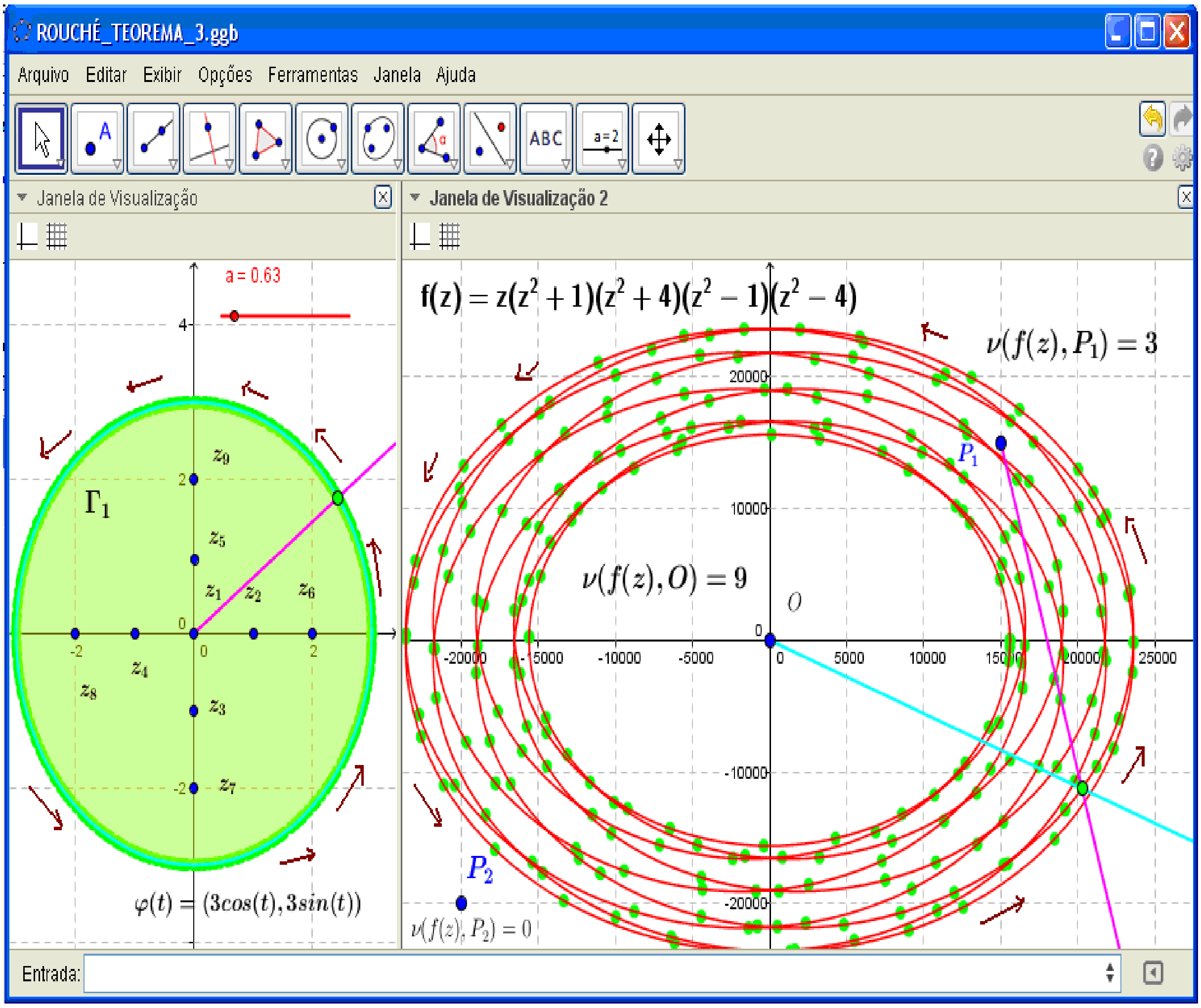

Figure 5. *The use of the winding number with intention to confirm the localization of the roots*

To conclude this section, certainly, the method proposed by Needham (2000, p. 341) that we have talked about earlier in this paper, it can become very complicated in the context of use of the software Geogebra. Thus, in the next section, we describe a way of counting the revolutions of a moving point on a trajectory indicated by $f(\Gamma)$, for a given point. This way becomes more simplies than scoring each intersection with signs $\oplus$ or $\otimes$ as we see in Figure 2. In this context, the symbol $f(\Gamma)$ symbol describes the action of an analytic function, when restricted to a parametric curve.

3. The use of Geogebra and the CAS Maple in the teaching

Certains tasks can become tedious when we disregard the potential of technology. For example, when we seek to determine the behavior and the analytic description for the real and imaginary parts of the $p(z) = (z-5)^5(z+2)^8(z-7)(z+6)$. This task can surprise us. In fact, in the figure below, we can imagine that is impossible determine the analitycal behavior of $\mathrm{Re}(p(x+iy))$ and $\mathrm{Im}(P(x+iy))$ whithout the computer´s help. We also highlight the example shown in Skakarchi (2000, p. 80) relatively to the polynomial $g(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$.

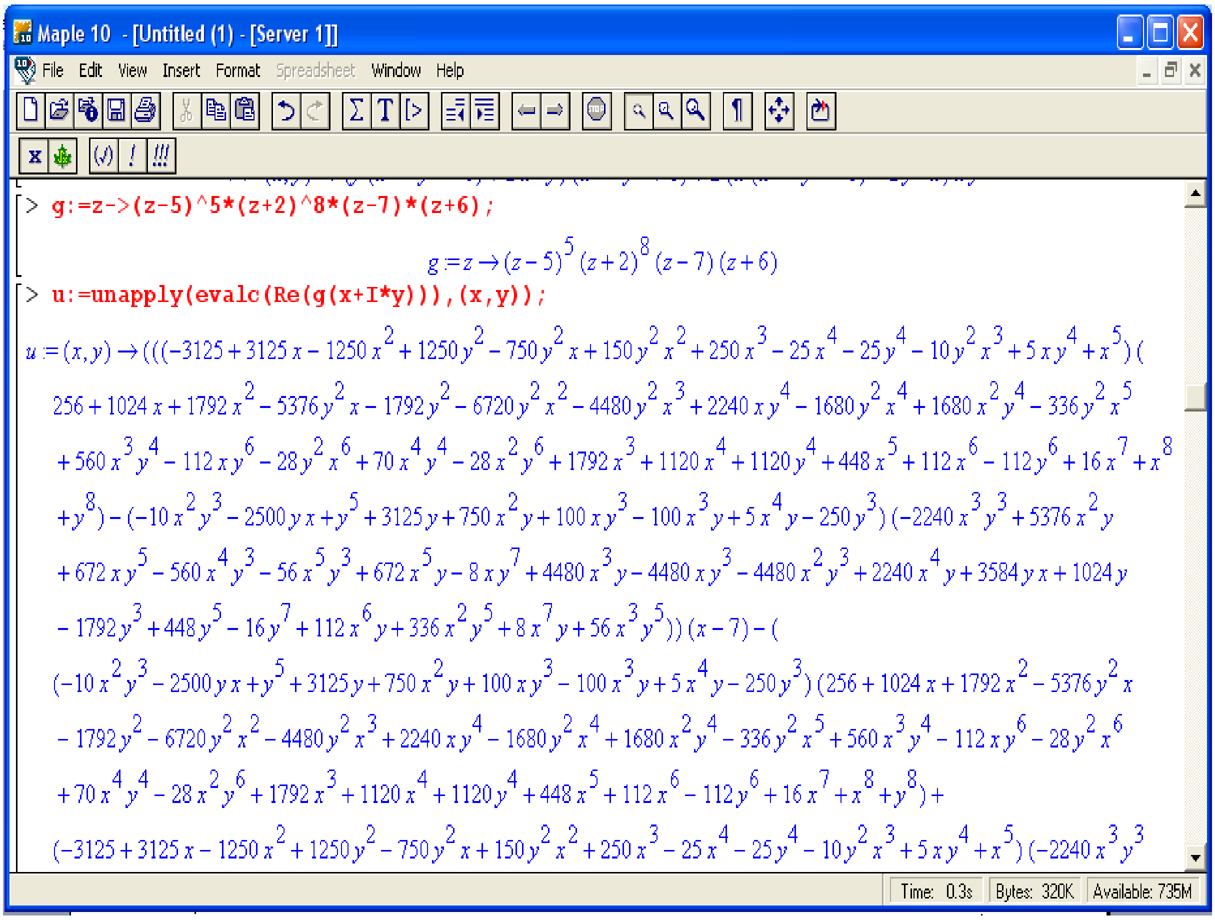

Figure 6. *The analithical expression of the real part related to the polynomial fonction* $g(z)$

Our first example is the polynomial equation $f(z) = z^7 + z^2 - 2 = \mathrm{Re}(f) + i\,\mathrm{Im}(f)$. With CAS Maple, we obtain the following analytical expressions $\mathrm{Re}(f) = -2 + x^2 - 21x^5y^2 + 35x^3y^4 - 7xy^6 + x^2 - y^2$, $\mathrm{Im}(f) = 7x^6y - 35x^4y^3 + 21x^2y^5 - y^7 + 2xy$. In the figure 7, we can observe the points which are the curvas level intersection indicated by $\mathrm{Re}(f) \cap \mathrm{Im}(f) = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$. In the figure below (on the right side), we visualize two neighborhoods (Γ_1 and Γ_2). In the ring $1 \le |z| \le 2$ we indentify all topological position for the polynomial roots. We verified also the existence of seven points in the disk $|z| \le 2$. Our method emphasizes the graphic-geometrical behavior of the all conceptual elements related in the study of polynomial roots. In figure 8, we show some restrictions of the surfaces described by $(x, y, \mathrm{Re}(f)) \in IR^3$ and $(x, y, \mathrm{Im}(f)) \in IR^3$ and its level curves. From these graphics, we will understand the meaning and the elements wich we indicate with the software Geogebra. From this scene in the figures 7 and 8, we conjecture acquire the numerical feeling about the roots.

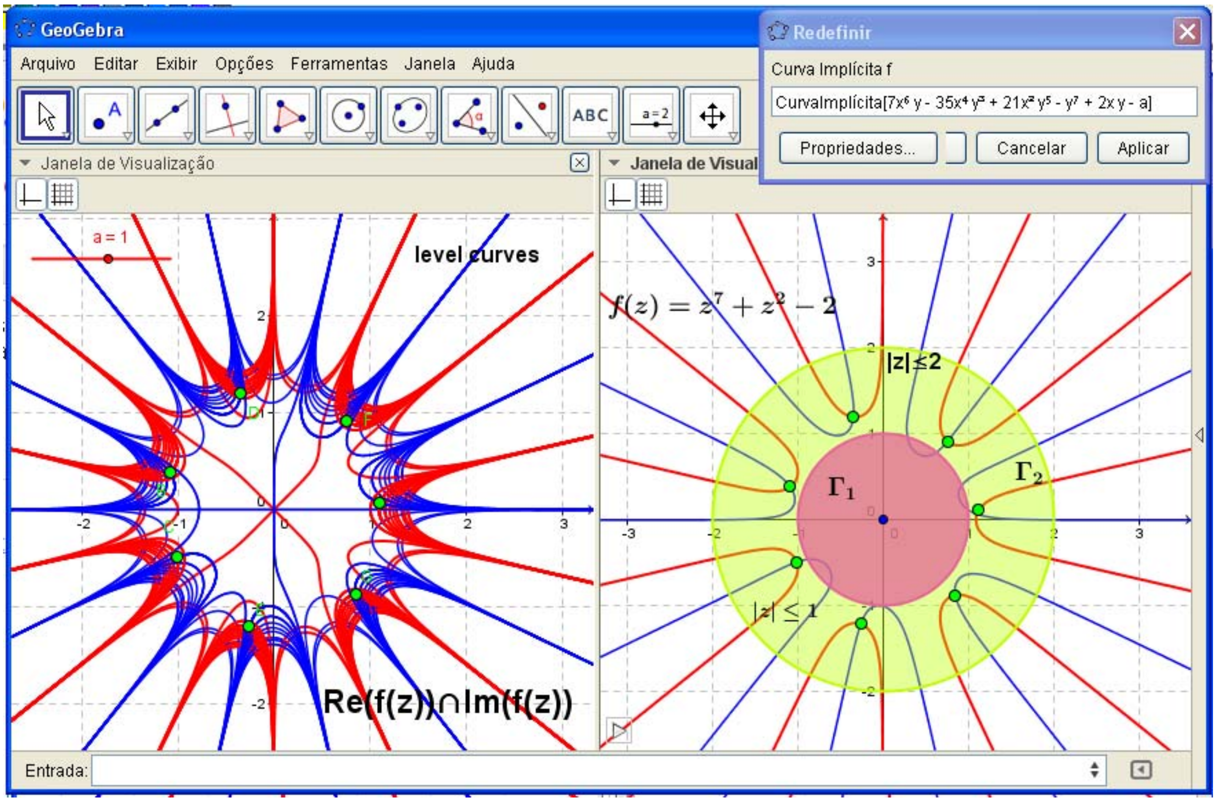

Figure 7. *Visualization of the region in the plane which we expected to find the roots*

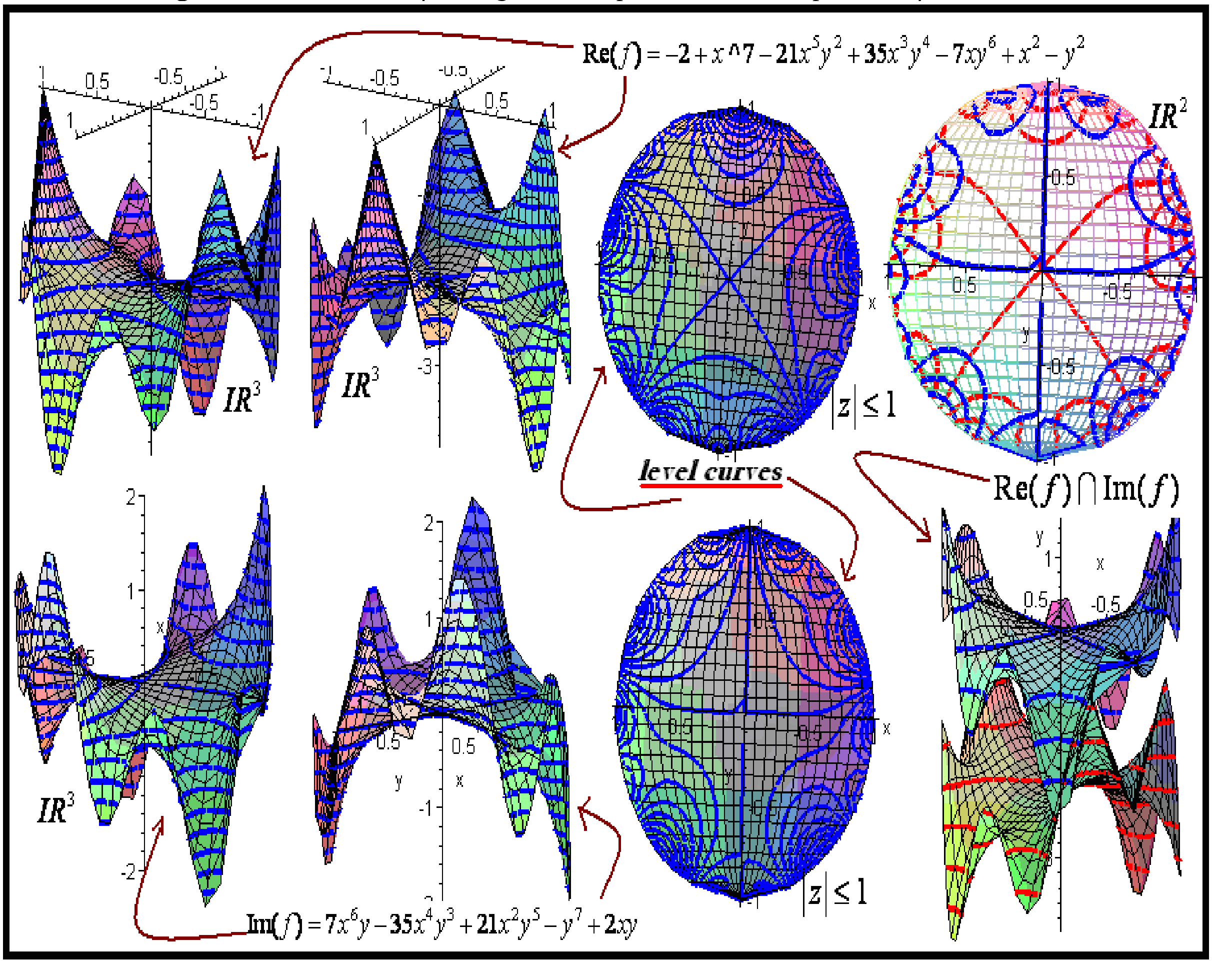

Figure 8. *The real and imaginary parts of the complex variable function in the* IR^2 *and* IR^3

From this preliminary scenario with the use of both softwares, we will employ another theorem that confirms our conjectures about the figures 7 and 8. In fact, we will use the Rouché´s theorem. We consider now the fonction $f(z) = f(x+iy) = z^4 - 5z + 1 = \mathrm{Re}(f(z)) + i\,\mathrm{Im}(f(z))$. Traditionally, we take the restriction $|z| \leq 1$. We still indicate the behavior of the level curves in the plane. We see that these curves are ortogonaly. On the other hand, we observe the graphical-geometric behavior of the real and imaginary parts. When we analyze the behavior of the (blue and greend) level curves, we can conclude that we deal with harmonic functions, since the level curves relating to real and imaginary parts are self intersecting orthogonally in the interested region indicated in Cartesians coordinates $x^2 + y^2 \leq 1$.

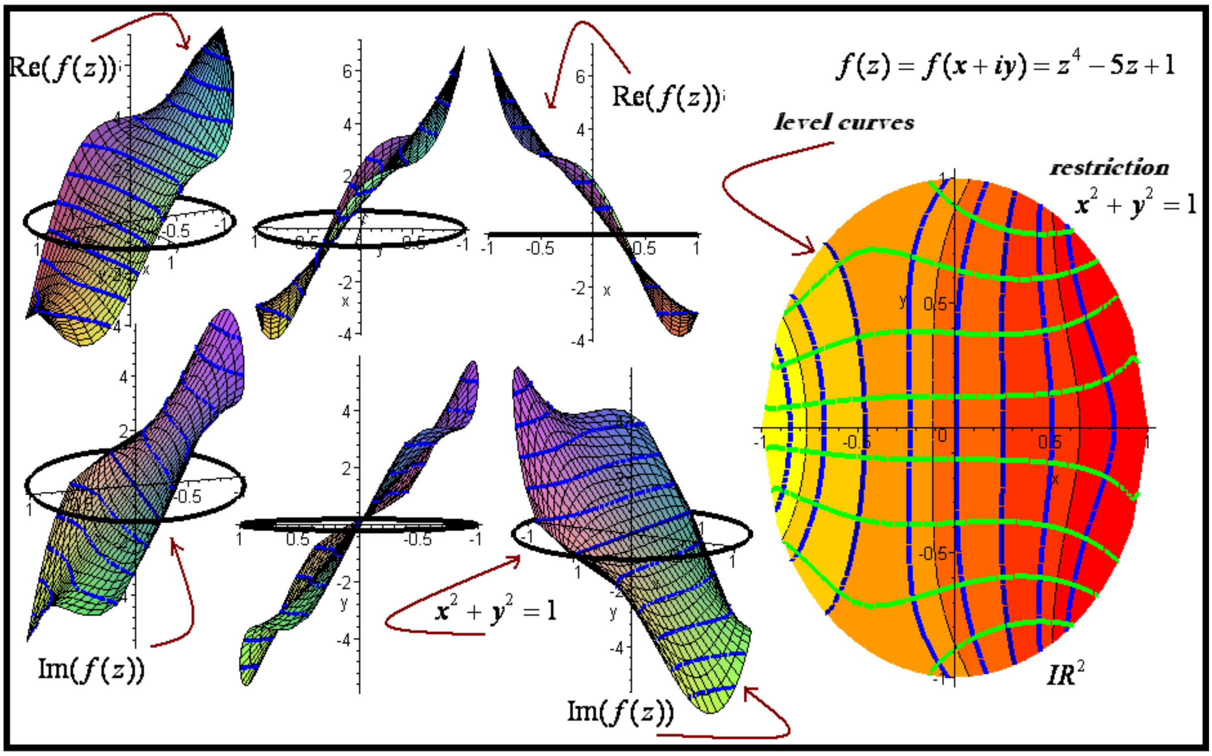

Figure 9. *The graphic-geometric behavior of the restrictions for the function and its real and imaginary parts*

In the figure 10, we take the fonction $f(z) = z^4 - 5z + 1 \in □[x]$. According to TFA, we can declare that exist exactly four roots for this function! On the other hand, this information does not explain anything about the location and qualitative properties of the roots. For example, if there is any root with multiplicity greater than one. With the Geogebra, we use a disk wich can varies the diameter ($0 \leq r \leq 2$).Differently proposed by Needham, we count the revolutions executed by moving points on the curve below (four revolutions in this case). We still point the trace described by a parameterized moving point P which is moving in a counterclockwise direction (the arrows indicate the direction).

With this construction, a period corresponding to $0 \mapsto 2\pi$ we count the number of turns around of the points E and O. Aftter that, we conclude that $\nu[f(z), E] = 2$ and $\nu[f(z), O] = 4$. This numerical data tell us that the all roots of $f(z) = z^4 - 5z + 1$ are in the circle $|z| \leq 2$. However, by virtue de Rouché´s theorem, we can get the $p(z) = z^4 + 1$ and $g(z) = -5z$. We write the inequality $|p(z)| = |z^4 + 1| \leq 2 < 5 = |g(z)|$. Due the Rouché´s theorem, $p(z) + g(z) = f(z)$ and $g(z) = -5z$ has the same number of the zeros in the unit circle (figure 11). From this fact, we indicated in the figure 10, the other roots $\{z_2, z_3, z_4\}$ in the ring $1 \leq |z| \leq 2$. By the TFA and the Rouché´s theorem, the last root is in the unit circle!

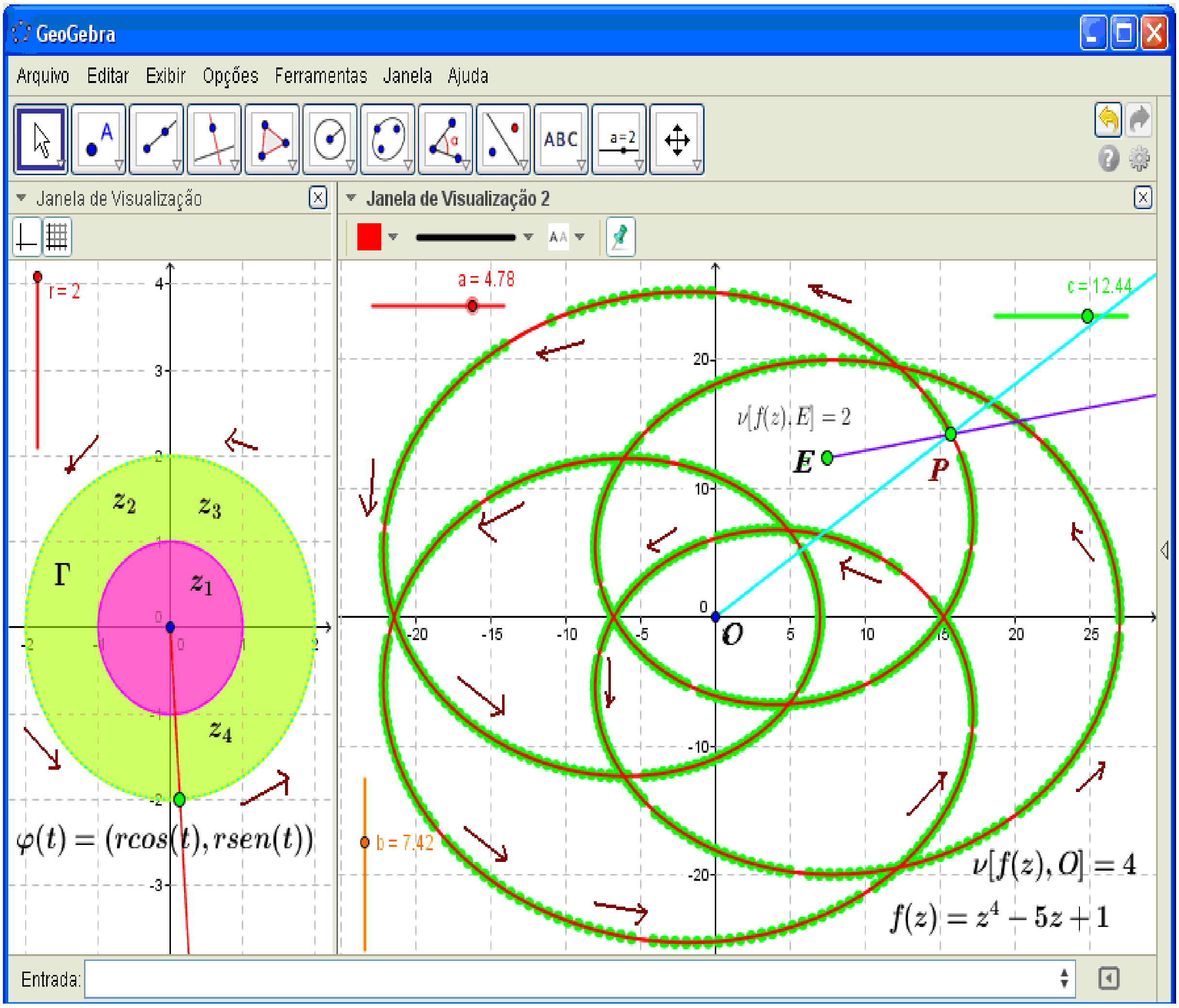

Figure 10. *Visual computing of the winding number related to the all polynomial roots with* $|z| \le 2$

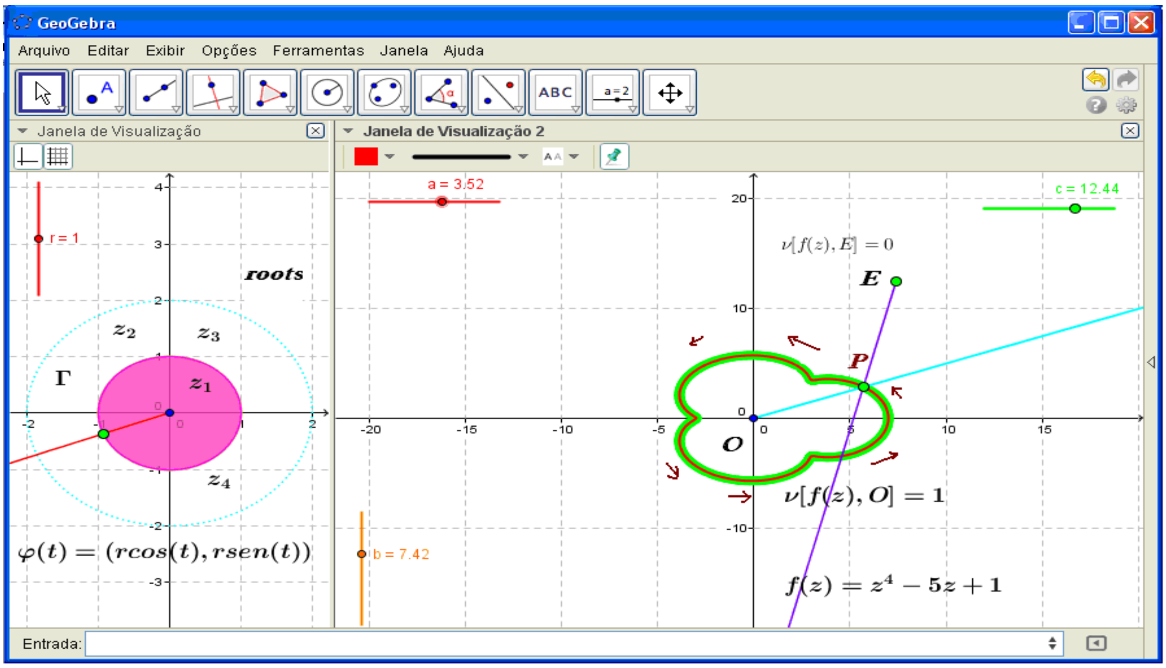

Figure 11. *Visual computing of the winding number related to some of polynomial roots with* $|z| \le 1$

In the figure below, we describe a curve that gradually deforms around the origin of the □ $- plane$. However, on the left side, while the unit disk involves a single root, we noticed that the number of turns corresponding to value is preserved $\nu[f(z),O]=1$. On the other hand, at some point, as the action of the deformation, the disk centered at the origin will no longer engage the root. In fact, at the same time, when we consider the right side of the figure, we put emphasis on the change of color to mark the time when the curve shows variation in the number of turns around the origin. In the region that we see the pink curves, we visualized that the curve can not wind around the origin O. Therefore, we can expect that $\nu[f(z),O]=0$.

In the figure 12, with the software Geogebra, we can deform continuously the red loop L in the orange loop L, since the winding number is always $\nu[f(z),O]=1$. On the other hand, when we observe that the winding number changes, we can verify too that the number of the roots of the $f(z)=f(x+iy)=z^4-5z+1$ inside the disk Γ changes (on the left side). We can visualize these modifications in the figure 12. This graphic-geometric scenario can promote our conceptual understanding of the Rouché´s theorem and the A.P.

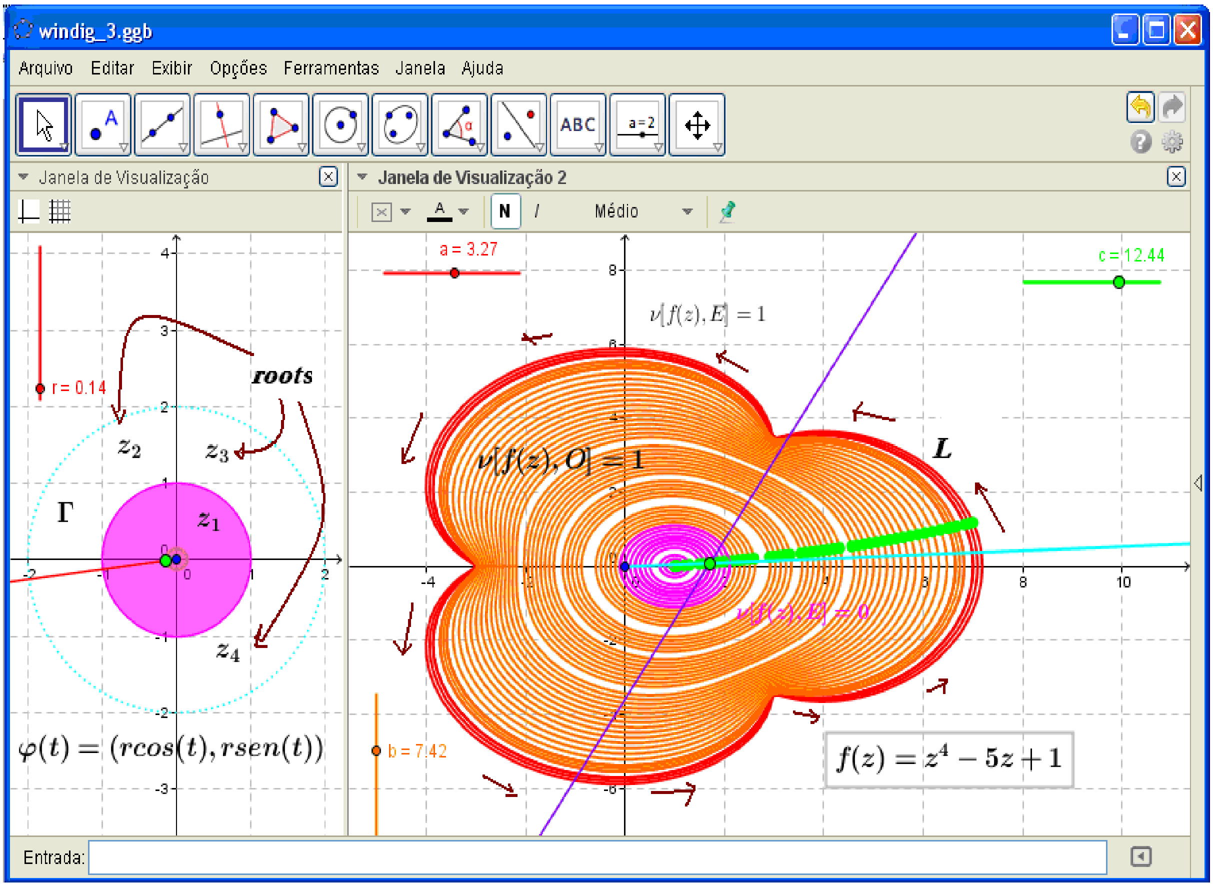

Figure 12. *We can observe the change values related to the "winding number"*

Finally, our previous argument was based on the following principle stated by Needham (2000, p. 350): "the total number $p-$points inside Γ (counted with their topological multiplicities) is equal to the winding number of $h(\Gamma)$ round p". Certainly, this article does not intend to present a more detailed discussion of such strong property of topological nature, however, we point out a heuristic route to its didactic transmission based on the technology for the teaching context of CA.

4. Conclusion

We recognize the effort of this author in other to convey the reader about some complex and conceptual mathematical ideas related to the geometric and topological notions. However, we can not disregard the static character of the figures employed by Needham (2000). In particular, the TFA shows reasonable abstractionist aspect. Furthermore, other theorems and the Argument Principe can also be used with the intention to verify the location of the roots of a function $f(z) = f(x+iy)$.

In this work we combined the view point of Needham (2000) with the current use of technology. In particular, we showed the way to explore the both softwares with the purpose to analyse qualitatively the polynomial roots. We still detache some limitations for the both softwares. In fact, we find with the CAS Maple enormous analytical expressions that become to be unworkable being investigated with the dynamic system Geogebra (see figure 13). These expressions are in Cartesian coordinates. Thus, to express them as level curves in the Geogebra, we have to introduce a parameter.

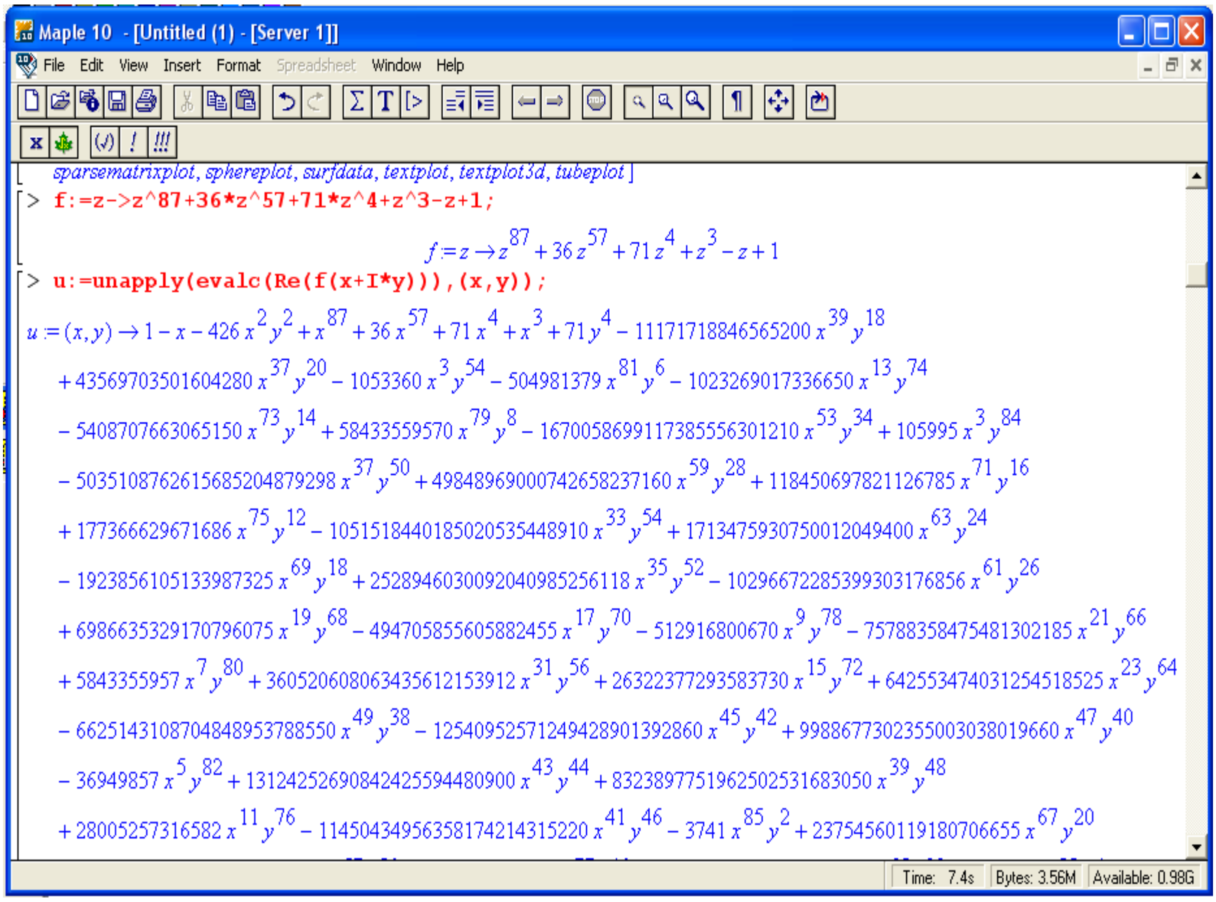

Figure 13. *The CAS permits to find analythical expressions related to the real and imaginary parts of a complex polynomial fonction*

Needham (2000, p. 341) also points out another non-trivial problem that can be studied with the software Geogebra. Indeed, from the Mathematical Analysis, we know that the Hopf theorem can be generalized to any n-dimensional space. In his book, Needham (2000, p. 341) mentioned that "a loop K may be continuously deformed into another loop L, without ever crossing the point p, if only if K and L have the same winding number around p". This and other issues will be our interest in the future works in the context of the Visual Complex Analysis ans its teaching.

To conclude this article, we indicate the main elements employed here, with the help of both softwares. So, our systematic method is characterized by the following steps:

(i) analytical description of the real and imaginary parts of the function $f(z)$ with the CAS Maple - $(x, y, \mathrm{Re}(f)) \in IR^3$, $(x, y, \mathrm{Im}(f)) \in IR^3$;

(ii) graphical description of the real and imaginary parts of the function $f(z)$ with the CAS Maple in the IR^3. In this step we determine the set Γ.

(iii) choice of a convenient parameterization $\alpha(t) = (r \cdot \cos(t), r \cdot \sin(t))$ for the Γ in the $\mathbb{C} - plane$ with Geogebra;

(iv) description of the set $f(\Gamma)$ according to the parameters indicated in the step (iii);

(v) use ot the Rouché´s theorem to determine a suitable neighborhood $|z| \leq r$ in the $\mathbb{C} - plane$ that encircle the all or some of the roots for the equation $f(z) = 0$;

(vi) counting and determing of the winding number $\frac{1}{2\pi i}\oint_{\alpha} \frac{dz}{z-P} = \nu[f(z), P] = n \in \mathbb{Z}$ (Krantz, 2007, p. 139) relatively the each suitable neighborhood with intention to confirm the polynomial equation roots, where $\alpha(t) = (r \cdot \cos(t), r \cdot \sin(t))$, $t: 0 \mapsto 2\pi$ and $0 < r$.

(vii) determining the static behavior of the level curves with the CAS Maple or a dynamic behavior of the level curves relatively the set $f(\Gamma)$ with the software Geogebra.

(viii) Finally, the visual property and formal-logic confirmation provided by TFA.

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