ACTA DIDACTICA NAPOCENSIA
Volume 9, Number 4, 2016

THE BIVARIATE (COMPLEX) FIBONACCI AND LUCAS POLYNOMIALS: AN HISTORICAL INVESTIGATION WITH THE MAPLE´S HELP

Francisco Regis Vieira Alves, Paula Maria Machado Cruz Catarino

Abstract: The current research around the Fibonacci´s and Lucas´ sequence evidences the scientific vigor of both mathematical models that continue to inspire and provide numerous specializations and generalizations, especially from the sixthies. One of the current of research and investigations around the Generalized Sequence of Lucas, involves it´s polinomial representations. Therefore, with the introduction of one or two variables, we begin to discuss the family of the Bivariate Lucas Polynomias (BLP) and the Bivariate Fibonacci Polynomials (BFP). On the other hand, since it´s representation requires enormous employment of a large algebraic notational system, we explore some particular properties in order to convince the reader about an inductive reasoning that produces a meaning and produces an environment of scientific and historical investigation supported by the technology. Finally, throughout the work we bring several figures that represent some examples of commands and algebraic operations with the CAS Maple that allow to compare properties of the Lucas´polynomials, taking as a reference the classic of Fibonacci´s model that still serves as inspiration for several current studies in Mathematics.

Key words: Lucas Sequence, Fibonacci´s polynomials, Historical investigation, CAS Maple.

1. Introduction

Undoubtedly, the Fibonacci sequence preserves a character of interest and, at the same time, of mystery, around the numerical properties of a sequence that is originated from a problem related to the infinite reproduction of pairs of rabbits. On the other hand, in several books of History of Mathematics in Brazil and in the other countries (Eves, 1969; Gullberg, 1997; Herz, 1998; Huntley, 1970; Vajda, 1989), we appreciate a naive that usually emphasizes eminently basic and trivial properties related to this sequence. This type of approach can provide a narrow and incongruent understanding of the Fibonacci sequence, mainly about it´s current evolutionary stage.

On the other hand, from the work of the mathematicians François Édouard Anatole Lucas (1842 – 1891) and Gabriel Lamé (1795 – 1870), we observe a progressive return by the interest of the study of numerical sequences and their properties. Thus, we highlight the following set of the numerical sequences: $\{1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,6765,10946,\ldots,f_n,\ldots\}$; $\{1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,2.207,3571,5778,9349,15127,\ldots,\mathrm{L}_n,\ldots\}$; $\{1,2,5,12,29,70,168,408,,\ldots,\mathrm{P}_n,\ldots\}$; $f_{n+1}=f_n+f_{n-1}$; $L_{n+1}=L_n+L_{n-1}$; $P_{n+1}=2P_{n-1}+L_{n-2}$, $n\geq 1$.

We observed that the initial values are indicated by $f_1=1,\ f_2=1,\mathrm{L}_1=,\mathrm{L}_2=3,\mathrm{P}_1=1,\mathrm{P}_2=2$. The Fibonacci polynomials were first studied in 1883 by Belgian mathematician Eugene Charles Catalan (1814 - 1894) and German mathematician Ernest Erich Jacobsthal (1881 - 1965). Thus, Catalan defined the following family of Fibonacci polynomial functions as follows.

Definition 1: We will call the Fibonacci Polynomial Sequence - SPF, the set of polynomial functions described by the recurrence relation $f_1(x)=1,\ f_2(x)=x,\ f_n(x)=x\cdot f_{n-1}(x)+f_{n-2}(x)$, $n\geq 1$.

Undoubtedly, through the previous definition, we can perceive the generalization process that initially occurred with the Fibonacci sequence and, gradually, after a few decades, began to be registered in

Received December 2016.

several other generalized sequence models as well. Thus, in the following section, we will study the recursive relations arising from the introduction of two real variables, even when we consider the introduction of an imaginary unit. Such mathematical definitions are relatively recent and reflect an evolutionary mathematical and epistemological process of the Fibonacci and Lucas´ sequence.

2. The Bivariate (Complex) Fibonacci and (Complex) Lucas Polynomials

The scientific interest in returning study and research around the Fibonacci and Lucas sequences can be derived from the inauguration and publication of the periodical entitled *The Fibonacci Quarterly*, in 1963 (Gullberg, 1997, p. 243; Posamentier & Lehmann, 2007, p. 28; Stakhov, 2009, p. 130). From this fact, there was a greater publicity and dissemination of information and scientific data related to a diversity of repercussions and progressive specializations in the research subfields of the Lucas and Fibonacci model (Brousseau, 1965, 1967; Gould, 1981; Honsberger, 1985; Livio, 2002; Vajda, 1989; Walser, 2001).

As we mentioned in the previous section, one of the forms of specialization of the Fibonacci model occurred from the introduction of a kind of parameter or variable and that gave rise to the definition of the Fibonacci polynomial sequence (see definion 1). Similarly, from the seventies, a similar kind of generalization of the Lucas´ model became better know.

For example, Asci & Gurel (2012, p. 1) mention that the greater interest in the study of the Lucas polynomials in 1970 occurred with the work of Bicknell (1970) and Hoggat & Bicknell (1973a; 1973b). In addition, for it´s formulation, we recorded the following recurrence.

Definition 2: We will call the Lucas Polynomial Sequence - SPL, the set of polynomial functions described by the recurrence relation

$L_0(x) = 2, \mathrm{L}_1(x) = x, L_n(x) = x \cdot L_{n-1}(x) + L_{n-2}(x)$. (Bicknell, 1970).

From a tradition of works (Hoggat & Long, 1974; Swamy, 1968; Web & Parberry, 1969) that discussed properties of the bivariate polynomials, we still have the following definition.

Definition 3: The Generalized Bivariate Fibonacci Polynomial (GBFP) may be defined as $H_0(x,y) = a_0, \mathrm{H}_1(x,y) = a_1, \mathrm{H}_{n+1}(x,y) = x \cdot H_{n-1}(x,y) + y \cdot H_{n-2}(x,y), n \geq 1$. We assume $x, y \neq 0, x^2 + 4y \neq 0$. (Catalini, 2004a; 2004b).

From a particular cases, taking $a_0 = 0, a_1 = 1$ and $a_0 = 2, a_1 = x$ we can define.

Definition 4: The Bivariate Fibonacci Polynomial (BFP) may be defined as $F_0(x,y) = 0, F_1(x,y) = 1, F_{n+1}(x,y) = x \cdot F_{n-1}(x,y) + y \cdot F_{n-2}(x,y), n \geq 1$.

Definition 5: The Bivariate Lucas Polynomial (BLP) may be defined as $L_0(x,y) = 2, \mathrm{L}_1(x,y) = x, \mathrm{L}_{n+1}(x,y) = x \cdot L_{n-1}(x,y) + y \cdot L_{n-2}(x,y), n \geq 1$.

We introduce the unity imaginary ‘i’ in order to define the following sequences.

Definition 6: We will call the Bivariate Complex Fibonacci Polynomial Sequence (BCFP) described by the recurrrente relation $F_0(x,y) = 0, \mathrm{F}_1(x,y) = 1, F_{n+1}(x,y) = ix \cdot F_n(x,y) + y \cdot F_{n-1}(x,y), n \geq 1$.

In the similar way, Asci & Gurel (2012) bring a last definition

Definition 7: We will call the Bivariate Complex Lucas Polynomial Sequence (BCLP) described by the recurrrente relation $L_0(x,y) = 2, \mathrm{L}_1(x,y) = ix, L_{n+1}(x,y) = ix \cdot L_n(x,y) + y \cdot L_{n-1}(x,y), n \geq 1$.

From the previous mathematical definitions, we can easily determine the polynomial forms (BFP, BLP, BCFP, BCLP) we can see in the figure below. Clearly, we note the presence of the imaginary unity of algebraic expressions that tend to become increasingly complex, as the order of the corresponding element increases. Thus, similarly to the historical elements that acted as epistemological obstacles to the establishment of an explicit formula of the Fibonacci and Lucas terms (Spickerman, 1982; Waddill & Sacks, 1967; Witford, 1977), we may express a similar interest for the determination of an explicit formula of the terms presented in the bivariate polynomial sequence.

On the other hand, although we find the effort of some authors to formulate certain algebraic representations that may facilitate the determination of particular elements $F_n(x,y)_{n\in IN}$ and $L_n(x,y)_{n\in IN}$ of any order present in the sequence, we also find the high operational cost conditioned by the use of high order matrix representations. Consequently, when we explore the computational model we can anticipate and test certain properties and the validity of certain mathematical theorems. In this case, Asci and Gurel (2012) comment the first elements present in the Bivariate Complex Fibonacci and Lucas´ family, and then in Figure 2, we can observe the proposition of a matrix, of order 'n', that allows determining explicitly any of it´s elements.

n	$F_n(x,y)$		n	$L_n(x,y)$
0	0		0	2
1	1		1	ix
2	ix		2	$-x^2+2y$
3	$-x^2+y$		3	$-x^3i+3xyi$
4	$-x^3i+2xyi$	and	4	$x^4-4x^2y+2y^2$
5	$x^4-3x^2y+y^2$		5	$x^5i-5x^3yi+5xy^2i$
6	$x^5i-4x^3yi+3y^2xi$		6	$-x^6+6x^4y-9x^2y^2+2y^3$
7	$-x^6+5x^4y-6x^2y^2+y^3$		7	$-x^7i+7x^5yi-14x^3y^2+7xy^3i$
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Figure 1. *Ascii & Gurgel (2012) list the first terms of the Polynomial Bivariate Fibonacci and Lucas sequence.*

$$D_n(x,y)=\begin{bmatrix}1 & 1 & 0 & \cdots & 0\\ 0 & ix & 1 & \ddots & \vdots\\ 0 & -y & ix & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & 0 & -y & ix\end{bmatrix},\ n\geq 1 \quad H_n(x,y)=\begin{bmatrix}2 & 1 & 0 & \cdots & 0\\ 0 & \frac{ix}{2} & 1 & \ddots & \vdots\\ 0 & -y & ix & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & 0 & -y & ix\end{bmatrix},\ n\geq 1$$

Figure 2. *Ascii & Gurgel (2012) indicate the matrices whose determinants produce the elements of both sequences*

In a precise way, on the left side, the determinant of the matrix allows to ascertain any element of the type $F_n(x,y)=\det(D_n(x,y))$, for $n\geq 1$, while on the right side, through the determinant of the indicated matrix, we can make explicit from the formula $L_n(x,y)=\det(H_{n-1}(x,y))$, $n\geq 1$ (see

Figure 2). Certainly, when we deal with matrices of fourth, fifth, and sixth and N-th orders, we will have serious operational difficulties and, in view of this, the necessity of using the computational model is unquestionable. Before, however, we will see in the next section some elemental properties of divisibility related to the Lucas´s sequence, that we have indicated by $\{L_n\}_{n\in IN}$.

3. Some preliminarly properties

Jeffery & Pereira (2014) take the particular matrix $A=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$ and $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$. Thus, by induction, it will be easy to determine that $A\cdot Q^n=\begin{pmatrix}L_{n+1} & L_n\\ L_n & L_{n-1}\end{pmatrix}=Q^n\cdot A$, for $n\geq 1$. On the other hand, we still observe $A=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}=2\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=2Q-I$. Moreover, we have the following identities $A^2=5I$, $A\cdot Q=Q\cdot A=\begin{pmatrix}3 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}L_3 & L_2\\ L_2 & L_1\end{pmatrix}$ (commutative property).

Lemma 1: Let $k,n\in IN$, with k odd, then $L_n \setminus L_{k\cdot n}$.

Proof. Preliminary, we take $AQ^n=\begin{pmatrix}L_{n+1} & L_n\\ L_n & L_{n-1}\end{pmatrix}$ and, taking the determinant of the same expression, we obtain $\det(AQ^n)=\det\begin{pmatrix}L_{n+1} & L_n\\ L_n & L_{n-1}\end{pmatrix}=L_{n+1}L_{n-1}-L_n{}^2$. But we will also have to $\det(AQ^n)=\det(A)\det(Q^n)=-5\cdot\det(Q)\cdots\det(Q)=5\cdot(-1)^{n+1}$. Now, if we consider that, for a particular value $'n_0'$ we have the propertie $5\setminus L_{n_0}$ and, but in view of $5\setminus L_{n_0+1}L_{n_0-1}-L_{n_0}{}^2$, we could conclude that $5\setminus L_{n_0+1}L_{n_0-1}\therefore 5\setminus L_{n_0-1}$. Finally, we must verify that $5\setminus L_{n_0+1}=L_{n_0}+L_{n_0-1}$. However, this can not occur. Thus, no element present in the Lucas sequence is divisible by 5. On the other hand, since k is odd, we can write $k=2m+1$. Now, we observe that the matrix $A\cdot Q^n=\begin{pmatrix}L_{n+1} & L_n\\ L_n & L_{n-1}\end{pmatrix}$ is a diagonal matrix $\bmod L_n$. Morover, we get the matrix $AQ^n\cdots AQ^n=(AQ^n)^k$ is also a diagonal matrix $\bmod L_n$. In addition, we still observe that holds $(AQ^n)^k=(AQ^n)^{2m+1}=A^{2m+1}Q^{nk}=(A^2)^m A\cdot Q^{nk}=(5^m I)A\cdot Q^{nk}=5^m A\cdot Q^{nk}$. From this, we obtain the identity $5^m\cdot\begin{pmatrix}L_{nk+1} & L_{nk}\\ L_{nk} & L_{nk-1}\end{pmatrix}=5^m\cdot AQ^{nk}=(AQ^n)^k$ is also a diagonal matrix $\bmod L_n$. Since L_n is not divisible by 5, we must have $L_n\setminus L_{nk}$, with k odd.

Lemma 2: Let $k,n\in IN$, with k even, then $L_n\setminus L_{k\cdot n}$.

Proof. In a similar way, we record $(AQ^n)^k=(AQ^n)^{2m}=A^{2m}Q^{nk}=(A^2)^m\cdot Q^{nk}=(5^m I)\cdot Q^{nk}=5^m\cdot Q^{nk}$. From this, we observe $5^m\cdot\begin{pmatrix}L_{nk+1} & L_{nk}\\ L_{nk} & L_{nk-1}\end{pmatrix}=5^m\cdot Q^{nk}=(AQ^n)^k$. Again, we repeat the argument that, we observe that the matrix $AQ^n=\begin{pmatrix}L_{n+1} & L_n\\ L_n & L_{n-1}\end{pmatrix}$ is a diagonal matrix $\bmod L_n$. Morover, we get the matrix $5^m \cdot Q^{nk} = (AQ^n)^k$ is also a diagonal matrix $\mod L_n$. Follow the result, Since L_n is not divisible by 5, we must have $L_n \setminus L_{nk}$, with k even.

Theorem 1: Let $\mathrm{m}, n \in IN$ and let $d = \gcd(m,n)$. If the numbers $\frac{m}{d}, \frac{n}{d}$ are both odd, then $\gcd(L_m, L_n) = L_d$.

Proof. From the lemma 1 and 2, by definition we know $d \setminus m, d \setminus n$ and, consequently, we have $L_d \setminus L_n$ and $L_d \setminus L_m$. Now, we show that $\gcd(L_m, L_n) \setminus \mathrm{L}_d$. If we consider the particular case $\gcd(L_m, L_n) = 1 \setminus \mathrm{L}_d$ it´s done. So we suppose $\gcd(L_m, L_n) = d > 1$. We know some basic properties that we can get the intergers numbers a, b such that $am + bn = \gcd(m,n)$ (Koshy, 2007; Tattersall, 2005). We still observe that if 'a' is odd, the other must be even (b is even), or reciprocally, in view of the properties of the G.C.D.. So, we take the matrices AQ^m, AQ^n are both diagonal matrices $\mathrm{mod}[\gcd(L_m, L_n)]$. On the other hand, we consider the matrices $\left(AQ^m\right)^a, \left(AQ^n\right)^b$ and the matricial product indicated $\left(AQ^m\right)^a \cdot \left(AQ^n\right)^b = A^a Q^{am} \cdot A^b Q^{bn} = A^a \cdot A^b Q^{am+bn}$. We record that $a = 2p+1, b = 2q$ and, follows that $\left(AQ^m\right)^a \cdot \left(AQ^n\right)^b = (A^2)^p \cdot A \cdot (A^2)^q Q^{am+bn} = (5I)^p A (5I)^q) \cdot Q^d$ $= 5^{p+q} \cdot (A \cdot Q^d)$. Finally, we still observe that $\left(AQ^m\right)^a \cdot \left(AQ^n\right)^b = 5^{p+q} \cdot (A \cdot Q^d) = 5^{p+q} \cdot \begin{pmatrix} L_{d+1} & L_d \\ L_d & L_{d-1} \end{pmatrix}$.

But, the matrices AQ^m, AQ^n are both diagonal matrices $\mathrm{mod}[\gcd(L_m, L_n)]$ and, the same propertie we have for the both matrices $\left(AQ^m\right)^a, \left(AQ^n\right)^b$. Hence, $[\gcd(L_m, L_n)]$ divides L_d and $\gcd(L_m, L_n) = d$.

In this section we study properties involving the divisibility of the numbers present in the Lucas sequence. Naturally, similar properties are widely known related to the Fibonacci sequence. In addition, from a specialized literature, we also know that the properties of divisibility are preserved, since we deal with polynomial functions in one or in two variables. Surprisingly, we shall see that the same regularity is not expected for the set Fibonacci numbers and the Lucas polynomials. (see fig. 3).

F102 = 2^{3} * 1597 * 3571 * 6376021 ** 919 * 3469
F104 = 3 * 7 * 233 * 521 * 90481 ** 103 * 102193207
F105 = 2 * 5 * 13 * 61 * 421 * 141961 ** 8288823481
F106 = 953 * 55945741 ** 119218851371
F108 = 2^{4} * 3^{4} * 17 * 19 * 53 * 107 * 109 * 5779 ** 11128427
F110 = 5 * 11^{2} * 89 * 199 * 661 * 474541 ** 331 * 39161
F112 = 3 * 7^{2} * 13 * 29 * 47 * 281 * 14503 ** 10745088481
F114 = 2^{3} * 37 * 113 * 797 * 9349 * 54833 ** 229 * 95419
F116 = 3 * 59 * 19489 * 514229 ** 347 * 1270083883
F117 = 2 * 17 * 233 * 135721 ** 29717 * 39589685693
F118 = 353 * 2710260697 ** 709 * 8969 * 336419
F120 = 2^{5} * 3^{2} * 5 * 7 * 11 * 23 * 31 * 41 * 61 * 2161 * 2521 ** 241 * 20641
F126 = 2^{3} * 13 * 17 * 19 * 29 * 211 * 421 * 35239681 ** 1009 * 31249
F128 = 3 * 7 * 47 * 1087 * 2207 * 4481 ** 127 * 186812208641
F129 = 2 * 433494437 ** 257 * 5417 * 8513 * 39639893
F130 = 5 * 11 * 233 * 521 * 14736206161 ** 131 * 2081 * 24571
F132 = 2^{4} * 3^{2} * 43 * 89 * 199 * 307 * 9901 * 19801 ** 261399601
F134 = 269 * 116849 * 1429913 ** 4021 * 24994118449
F138 = 2^{3} * 137 * 139 * 461 * 829 * 18077 * 28657 ** 691 * 1485571
F140 = 3 * 5 * 11 * 13 * 29 * 41 * 71 * 281 * 911 * 141961 ** 12317523121

Figure 3. Dresel & Daykin (1965) studied the factorization of the Fibonacci´s numbers for n>100.

However, before we begin the next section, we recall the following property numerical behavior $f_{19} = 4181 = 37 \cdot 113$. Vernner & Hoggatt (1974, p. 113) mention that, in the Fibonacci sequence, the condition that ‘n’ be a prime is necessary but not sufficient for the primarity of the corresponding value. Moreover, the research about the primarity test relatively the both sequences has been developed for decades (Brillart; Montgomery & Silverman, 1988; Daykin & Dresel, 1970; Dubner & Keller, 1999). For example, Daykin & Dresel (1970, p. 30) observe that $L_{73} = 151549 \cdot 11899937029$, while 73 is a prime. In Figure 3, we can identify other cases wth the support of the technology. On the other hand, we will see that some desired properties can be verified in the case of the polynomial functions discussed in this work (Daykin & Dresel, 1970). (see Figure 3).

However, we record the definition 1, and the same authors declare that $f_p(x)$ is irreductible over the ring $Z[x]$ if only if ‘p’ is a prime. In addition, we have the following property over the ring $Z[x, y]$.

Lemma: If $p(x, y), q(x, y), r(x, y) \in Z[x, y]$, $p(x, y) \setminus q(x, y) \cdot r(x, y)$ and $p(x, y)$ is irreductible, then $p(x, y) \setminus q(x, y)$ or $p(x, y) \setminus r(x, y)$.

This will be discuss in the next section with the Maple´s help.

4. Some properties of the Bivariate Lucas Polynomial Sequence with the Maple´s help

We can mention pioneering works that discussed some divisibility criteria involving polynomial functions obtained by recurrence, present in the Fibonacci and Lucas sequence, in one or several variables (Bicknell, 1970; Hoggatt & Bicknell, 1973a; 1973b; Hoggatt & Long, 1974; Webb & Parberry, 1969). In particular, we note the results discussed Hoggatt & Long (1974). In fact, we enunciate the following results, in accordance the formal definition $F_0(x, y) = 0, F_1(x, y) = 1, F_{n+1}(x, y) = x \cdot F_n(x, y) + y \cdot F_{n-1}(x, y), n \geq 1$. With origin in these works of the seventies, we will announce some important results.

Theorem 2: For $m \geq 0, n \geq 0$ we have $F_{m+n+1}(x,y) = F_{m+1}(x,y)F_{n+1}(x,y) + \mathrm{y} \cdot F_m(x,y)F_n(x,y)$. (Hoggatt & Long, 1974, p. 114).

Proof. We observe that $F_2(x,y) = F_{0+1+1}(x,y) = x \cdot F_1(x,y) + y \cdot F_0(x,y) = F_{0+1}(x,y) \cdot F_{1+1}(x,y) + y \cdot F_0(x,y)F_1(x,y)$. Moreover, for $n = 2$ we write $F_3(x,y) = F_{1+1+1}(x,y) = x \cdot F_2(x,y) + y \cdot F_1(x,y) = x \cdot x + y \cdot 1 \cdot 1 =$ $= F_2(x,y)F_2(x,y) + \mathrm{y} \cdot F_1(x,y)F_1(x,y) = F_{1+1}(x,y)F_{1+1}(x,y) + \mathrm{y} \cdot F_1(x,y)F_1(x,y)$. For a fixed integer $m \geq 0$ we proceed for induction for 'n' and we assume that the propertie is true for $F_{m+n+1}(x,y) = F_{m+1}(x,y)F_{n+1}(x,y) + \mathrm{y} \cdot F_m(x,y)F_n(x,y)$. In the next step, we will consider the element $F_{m+n+2}(x,y) = x \cdot F_{m+n+1}(x,y) + y \cdot F_{m+n}(x,y) = x \cdot (F_{m+1}(x,y)F_{n+1}(x,y) + \mathrm{y} \cdot F_m(x,y)F_n(x,y)) + y \cdot F_{m+n}(x,y)$. But, we observe that $F_{m+n}(x,y) = F_{m+(n-1)+1}(x,y) = F_{m+1}(x,y)F_n(x,y) + \mathrm{y} \cdot F_m(x,y)F_{n-1}(x,y)$. Now, we take $xF_{m+1}(x,y)F_{n+1}(x,y) + x\,\mathrm{y} \cdot F_m(x,y)F_n(x,y) + y \cdot F_{m+1}(x,y)F_n(x,y) + \mathrm{y}^2 \cdot F_m(x,y)F_{n-1}(x,y) =$ $= [xF_{m+1}(x,y)F_{n+1}(x,y) + y \cdot F_{m+1}(x,y)F_n(x,y)] + [x\,\mathrm{y} \cdot F_m(x,y)F_n(x,y) + \mathrm{y}^2 \cdot F_m(x,y)F_{n-1}(x,y)] =$ $= F_{m+1}(x,y)[x \cdot F_{n+1}(x,y) + yF_n(x,y)] + \mathrm{y} \cdot F_m(x,y)[xF_n(x,y) + \mathrm{y} \cdot F_{n-1}(x,y)] = F_{m+1}(x,y)F_{(n+1)+1}(x,y) +$ $+ \mathrm{y} \cdot F_m(x,y)F_{n+1}(x,y)$. Finally, we obtained that: $F_{m+n+2}(x,y) = F_{m+1}(x,y)F_{(n+1)+1}(x,y) + \mathrm{y} \cdot F_m(x,y)F_{n+1}(x,y)$, for every $m \geq 0, n \geq 0$.

Lemma: For $n \geq 0$, then $\gcd(y, F_n(x,y)) = 1$. (Hoggatt & Long, 1974, p. 114)

Proof. The assertion is clearly true for $\mathrm{n} = 1 \therefore \gcd(y, F_1(x,y)) = \gcd(y,1) = 1$. Assume that it is true for any fixed $k \geq 1$. Then, since $F_{k+1}(x,y) = x \cdot F_k(x,y) + y \cdot F_{k-1}(x,y)$. If the condition does not hold, we could take $\gcd(y, F_{k+1}(x,y)) = d(x,y)$. But, en virtue the definion, we have $d(x,y) \backslash F_{k+1}(x,y), \mathrm{d(x,y)} \backslash \mathrm{y}$. Therefore, we will have $\mathrm{d(x,y)} \backslash F_{k+1}(x,y) - y \cdot F_{k-1}(x,y) = x \cdot F_k(x,y)$. However, it can not be $\mathrm{d(x,y)} \backslash F_k(x,y)$, since $\gcd(y, F_k(x,y)) = 1$. Therefore, we obtain that $\mathrm{d(x,y)} \backslash \mathrm{x}$ but, this can not occur unless that $\gcd(y, F_{k+1}(x,y)) = 1$.

Theorem 3: For $n \geq 0$, then $\gcd(F_n(x,y), F_{n+1}(x,y)) = 1$. (Hoggatt & Long, 1974, p. 116)

Proof. Again, the result is trivially true for $n = 0, n = 1$ since that $\gcd(F_0(x,y), F_1(x,y)) = \gcd(0,1)$ and $\gcd(F_1(x,y), F_2(x,y)) = \gcd(1,\mathrm{x}) = 1$. We assume that is true for $n = k-1$, where 'k' is a fixed interger $k \geq 2$ and we assume that $\gcd(F_k(x,y), F_{k+1}(x,y)) = d(x,y)$. Since we know $F_{k+1}(x,y) = x \cdot F_k(x,y) + y \cdot F_{k-1}(x,y)$. Again, by definition of the g.c.d., we will have the propertie $d(x,y) \backslash F_{k+1}(x,y) - x \cdot F_k(x,y) = y \cdot F_{k-1}(x,y)$. Thus, we will have $d(x,y) \backslash y \cdot F_{k-1}(x,y)$. But, we know that $\gcd(F_k(x,y), F_{k-1}(x,y)) = 1$ and, in this way, $F_{k-1}(x,y)$ is not divisible by $\mathrm{d}(x,y)$. Consequently, $d(x,y) \backslash y$ that is a irreductible polynomial in the variable 'y'. So, it´s a contradiction.

Theorem 4: For $m \geq 2$, then $F_m(x,y) \backslash F_n(x,y) \Leftrightarrow \mathrm{m} \backslash \mathrm{n}$. (Hoggatt & Long, 1974, p. 116)

Proof. $(\Rightarrow)$Again, we proceed by induction. Preliminarily, we observe that $F_m(x,y) \backslash F_{k \cdot m}(x,y)$ is true for $k = 1 \therefore F_m(x,y) \backslash F_{1 \cdot m}(x,y)$. We proceed by induction, for a fixed integer $k \geq 1$, that is, we know that $F_m(x,y) \backslash F_{k \cdot m}(x,y)$, for $k \geq 1$. In the next step, we will see if the following division occurs $F_m(x,y) \backslash F_{(k+1) \cdot m}(x,y)$. On the other hand, en virtue the identity of the theorem $F_{m+n+1}(x,y)=F_{m+1}(x,y)F_{n+1}(x,y)+\text{y}\cdot F_m(x,y)F_n(x,y)$, we write $F_{(k+1)m}(x,y)=F_{km+m}(x,y)=F_{km}(x,y)F_{m+1}(x,y)+\text{y}\cdot F_{km-1}(x,y)F_m(x,y)$. But, since we have (by hypothesis) $F_m(x,y)\backslash F_{k\cdot m}(x,y)$ and $F_m(x,y)\backslash F_m(x,y)$ immediately we obtain $F_m(x,y)\backslash F_{km}(x,y)\cdot F_{m+1}(x,y)+\text{y}\cdot F_{km-1}(x,y)\cdot F_m(x,y)$. Finally, we find that $F_m(x,y)\backslash F_{(k+1)\cdot m}(x,y)$. This propertie clearly implies that $F_m(x,y)\backslash F_{k\cdot m}(x,y)$, for $k\geq 1$. Thus, we still have $m\backslash n\Rightarrow F_m(x,y)\backslash F_n(x,y)$. $(\Leftarrow)$Now, for $m\geq 2$ we suppose $F_m(x,y)\backslash F_n(x,y)$ and we must obtain that $m\backslash \text{n}$. On the other hand, if we assume that 'n' is not divisible by 'm', by means of the division algorithm, exist intergers 'q' and 'r' with the condition $n=m\cdot q+r, 0<r\leq m$. Again, by the previous theorem, we take the formula $F_n(x,y)=F_{m\cdot q+r}(x,y)=F_{m\cdot q+(r-1)+1}(x,y)=F_{m\cdot q+1}(x,y)F_r(x,y)+\text{y}\cdot F_{m\cdot q}(x,y)F_{r-1}(x,y)$. Finally, we observe that $F_m(x,y)\backslash F_{m\cdot q}(x,y)$ by the first part of the proof and $F_m(x,y)\backslash F_n(x,y)$. Consequently, we still have $F_m(x,y)\backslash F_n(x,y)-\text{y}\cdot F_{m\cdot q}(x,y)\cdot F_{r-1}(x,y)=F_{m\cdot q+1}(x,y)\cdot F_r(x,y)$. But, since we know that $\gcd(F_{m\cdot q}(x,y),F_{m\cdot q+1}(x,y))=1$ and, the only possibility is $F_m(x,y)\backslash F_r(x,y)$ but it´s impossible, since we have the condition $0<r\leq m$, that is, the term $F_r(x,y)$ is a lower degree than $F_m(x,y)$. Therefore, $r=0$ and $m\backslash n$ and the proof is complete.

Theorem 5: For $m\geq 0, n\geq 0$ we have $\gcd(F_m(x,y),F_n(x,y))=F_{\gcd(m,n)}(x,y)$. (Hoggatt & Long, 1974, p. 116)

Proof. Through elementary properties, we know that exist integers r and s, say, $r>0$ and $s<0$ such that $\gcd(m,n)=r\cdot m+s\cdot n\therefore r\cdot m=(m,n)-s\cdot n$. Thus, by theorem, we write $F_{r\cdot m}(x,y)=F_{(m,n)-s\cdot n}(x,y)$ $=F_{(m,n)+(-s)\cdot n}(x,y)=F_{(m,n)}(x,y)F_{(-s)\cdot n+1}(x,y)+y\cdot F_{(m,n)-1}(x,y)F_{(-s)\cdot n}(x,y)=F_{(m,n)}(x,y)F_{-s\cdot n+1}(x,y)+$ $y\cdot F_{(m,n)-1}(x,y)\cdot F_{-s\cdot n}(x,y)$. But, let $\text{d}(\text{x},\text{y})=\gcd(F_m(x,y),F_n(x,y))$ and, consequently we have $\text{d}(\text{x},\text{y})\backslash F_{r\cdot m}(x,y)$ and $\text{d}(\text{x},\text{y})\backslash F_{-s\cdot n}(x,y)$. From these properties, we still obtain that $\text{d}(\text{x},\text{y})\backslash F_{r\cdot m}(x,y)-y\cdot F_{(m,n)-1}(x,y)\cdot F_{-s\cdot n}(x,y)=F_{(m,n)}(x,y)\cdot F_{-s\cdot n+1}(x,y)$. Now, if we conclude that $\text{d}(\text{x},\text{y})\backslash F_{(m,n)}(x,y)$ the proof is complete. However, if occur $\text{d}(\text{x},\text{y})\backslash F_{-s\cdot n+1}(x,y)$, we must observe that $1=\gcd(\text{d}(\text{x},\text{y}),F_{-sn+1}(x,y))$. Otherwise, we would have $d'(x,y)=\gcd(\text{d}(\text{x},\text{y}),F_{-sn+1}(x,y))$ and, we still know $1=\gcd(F_{-sn}(x,y),F_{-sn+1}(x,y))$ and $\text{d}(\text{x},\text{y})\backslash F_{-s\cdot n}(x,y)$. Consequently, we can get that $\text{d}'(\text{x},\text{y})\backslash F_{-s\cdot n}(x,y)$ and $\text{d}'(\text{x},\text{y})\backslash F_{-s\cdot n+1}(x,y)$ and we must obligatorily have to $1=\gcd(\text{d}(\text{x},\text{y}),F_{-sn+1}(x,y))$. Finally, the property $\text{d}(\text{x},\text{y})\backslash F_{(m,n)}(x,y)$ is verified.

We will give some examples below in order to verify the expected behavior of some particular cases, according to the theorems we have just demonstrated in detail.

$F_{19}(x,y)=x^{18}+17x^{16}y+120x^{14}y^2+455x^{12}y^3+1001x^{10}y^4+1287x^8y^5+924x^6y^6+330x^4y^7$ $+45x^2y^8+y^9$(is irreductible over Z[x,y],p=19)

$F_{23}(x,y)=x^{22}+21x^{20}y+190x^{18}y^2+969x^{16}y^3+3060x^{14}y^4+6188x^{12}y^5+8008x^{10}y^6+6435x^8y^7$ $+3003x^6y^8+715x^4y^9+66x^2y^{10}+y^{11}$(is irreductible over Z[x,y],p=23)

$F_{31}(x,y)=x^{30}+29x^{28}y+378x^26y^2+2925x^{24}y^3+14950x^{22}y^4+53130x^{20}y^5+134596x^{18}y^6+$ $245157x^{16}y^7+319770x^{14}y^8+293930x^{12}y^9+184756x^{10}y^{10}+75582x^8y^{11}+18564x^6y^{12}+$ $2380x^4y^{13}+120x^2y^{14}+y^{15}$(is irreductible over Z[x,y],p=31)

Before finalizing the list of some important results, we will enunciate a last theorem that provides an important characterization only for the elements present in the Polynomial Fibonacci´s sequence.

Theorem 6: Let $r = r(x, y)$ be any polynomial in the variables 'x' and 'y'. If there exists a least positive interger 'm', such that $r(x,y)\backslash F_m(x,y)$, then $r(x,y)\backslash F_n(x,y) \Leftrightarrow \mathrm{m}\backslash \mathrm{n}$. (Hoggatt & Long, 1974, p. 117).

Proof. By the theorem 4, we know if $\mathrm{m}\backslash \mathrm{n} \Rightarrow F_m(x,y)\backslash F_n(x,y)$. So, we admit that exists a least positive interger 'm', such that $r(x,y)\backslash F_m(x,y)$ and, by transitivity, we conclude $r(x,y)\backslash F_n(x,y)$. Now, we suppose that $r(x,y)\backslash F_n(x,y)$ and yet 'n' is not divisible by 'm'. Then, by the Euclidean Algorithim, exist integers $q, s, 0 < s < m$ and $n = m\cdot q + s$. Again, by the theorem 2, we can write $F_n(x,y) = F_{m\cdot q+s}(x,y) = F_{m\cdot q+(s-1)+1}(x,y) = F_{m\cdot q+1}(x,y)F_s(x,y) + y\cdot F_{m\cdot q}(x,y)F_{s-1}(x,y)$. But, since $r(x,y)\backslash F_m(x,y)$, $r(x,y)\backslash F_{m\cdot q}(x,y)$ and $r(x,y)\backslash F_n(x,y)$. Consequently, by the last identity, we still have $r(x,y)\backslash F_n(x,y) - y\cdot F_{m\cdot q}(x,y)F_{s-1}(x,y) = F_{m\cdot q+1}(x,y)F_s(x,y)$. From this, follows that $r(x,y)\backslash F_{m\cdot q+1}(x,y)F_s(x,y)$ and we know $\gcd(F_{m\cdot q}(x,y), F_{m\cdot q+1}(x,y)) = 1$. Thus, the only alternative is $r(x,y)\backslash F_s(x,y)$, however is a crontradicion, since exists a least positive interger 'm', such that $r(x,y)\backslash F_m(x,y)$ and $0 < s < m$. So, 'n' is divisible by 'm' and the proof is complete.

Now, we will study some properties of the divisibility and factorization of the polynomial elements present in the Lucas sequence. We will see that it does not enjoy a similar algebraic behavior. Before, however, let us look at some properties of the matrices.

From the definition $\mathrm{L}_n(x,y) = x\cdot L_{n-1}(x,y) + y\cdot L_{n-2}(x,y), n \geq 1$ we can determine some particular initial forms: $\mathrm{L}_0(x,y) = 2, \mathrm{L}_1(x,y) = x, \mathrm{L}_2(x,y) = x^2 + 2y$, $\mathrm{L}_3(x,y) = x^3 + 3xy$, $\mathrm{L}_4(x,y) = x^4 + 4x^2y + 2y^2$, $\mathrm{L}_5(x,y) = x^5 + 5x^3y + 5xy^2$, ,..etc. Catalani (2002) define the matrix $A(x,y) = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix} = A$ and $B(x,y) = \begin{pmatrix} x^2+2y & y \\ xy & 2y \end{pmatrix} = B$. Immediately, we will have $AB =$

$\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}\cdot\begin{pmatrix} x^2+2y & y \\ xy & 2y \end{pmatrix} = \begin{pmatrix} x^3+3xy & x^2+2y \\ (x^2+2y)\cdot y & x\cdot y \end{pmatrix}$, $AB^2 = \begin{pmatrix} x^4+4x^2y+2y^2 & x^3+3xy \\ (x^3+3xy)\cdot y & (x^2+2y)\cdot y \end{pmatrix}$, $AB^3 =$

$\begin{pmatrix} x^5+5x^3y+5xy^2 & x^4+4x^2y+2y^2 \\ (x^4+2x^2y+2x^2y+2y^2)y & (x^3+3xy)y \end{pmatrix}$, $AB^4 = \begin{pmatrix} x^6+6x^4y+9x^2y^2+2y^3 & x^5+5x^3y+5xy^2 \\ (x^5+2x^3y+3x^3y+4xy^2+xy^2)y & (x^4+2x^2y+2x^2y+2y^2)y \end{pmatrix}$.

In the figure below, we can analise and conjecture a closed form for the product $AB^n, n \geq 1$. So, from the appreciation of some particular products, we can acquire a better understanding about the follow identity $AB^n = \begin{pmatrix} L_{n+2} & L_{n+1} \\ L_{n+1}\cdot y & L_n\cdot y \end{pmatrix}$, $n \geq 1$. And, we can still work with the inverse matrix.

On the other hand, we can still get the inverse $A^{-1}(x,y) = \begin{pmatrix} 0 & \dfrac{1}{y} \\ 1 & -\dfrac{x}{y} \end{pmatrix} = A^{-1}$ and $B^{-1}(x,y) =$

$\begin{pmatrix} \frac{2}{x^2+4y} & -\frac{x}{x^2+4y} \\ -\frac{xy}{x^2+4y} & \frac{x^2+2y}{x^2+4y} \end{pmatrix} = B^{-1}$. From this, we can explore the algebraic behavior of the following produts: $(AB)^{-1}, (AB)^{-2}, (AB)^{-3}, (AB)^{-4}, (AB)^{-5}, (AB)^{-6}, \ldots, (AB)^{-n}, \mathrm{n} \geq 1.$ Moreover, with the CAS, we can find $(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} -\frac{x}{x^2y+4y^2} & \frac{x^2+2y}{(x^2y+4y^2)y} \\ \frac{x^2+2y}{(x^2y+4y^2)} & -\frac{x^3+3xy}{(x^2y+4y^2)y} \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{L}_1(x,y)}{(x^2+4y)\,\mathrm{y}} & \frac{\mathrm{L}_2(x,y)}{(x^2+4y)\,\mathrm{y}^2} \\ \frac{\mathrm{L}_2(x,y)}{(x^2+4y)\,\mathrm{y}} & -\frac{\mathrm{L}_3(x,y)}{(x^2+4y)\,\mathrm{y}^2} \end{pmatrix}$,

$$(AB)^{-2} = B^{-2}A^{-2} = \begin{pmatrix} -\frac{x}{x^2y+4y^2} & \frac{x^2+2y}{(x^2y+4y^2)y} \\ \frac{x^2+2y}{(x^2y+4y^2)} & -\frac{x^3+3xy}{(x^2y+4y^2)y} \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{L}_1(x,y)}{(x^2+4y)\,\mathrm{y}} & \frac{\mathrm{L}_2(x,y)}{(x^2+4y)\,\mathrm{y}^2} \\ \frac{\mathrm{L}_2(x,y)}{(x^2+4y)\,\mathrm{y}} & -\frac{\mathrm{L}_3(x,y)}{(x^2+4y)\,\mathrm{y}^2} \end{pmatrix}.$$

In Figure 4, we can observe the behavior of the product of the matrices indicated earlier and, through some preliminary cases, by means of an inductive process, formulate it´s general term. Of course, we can understand that calculus becomes impractical without the use of technology.

Now let us look at some properties of divisibility and factorization of some of the polynomial functions into two variables, present in the Bivariate Lucas Sequence. In Figure 5, we considered a strong propertie that permits determine any term of the BPL. In the left side, we can see fourth order matrix and, in the right side, we have considered a fifth order matrix. We calculate that $\det(H_4(x,y)) = L_3(x,y) = x^3+3xy = x\cdot(x^2+3y) = L_1(x,y)\cdot(x^2+3y)$ and $\det(H_5(x,y)) = L_4(x,y) = x^4+4x^2y+2y^2$.

We observed that the polynomial in two variables $x^4+4x^2y+2y^2$ is irreductible, while we found $L_3(x,y) = L_1(x,y)\cdot(x^2+3y)$. Moreover, we have $\det(H_9(x,y)) = L_8(x,y) = x^8+8x^6y+20x^4y^2+16x^2y^3+2y^4$ is a another irreductible polynomial over the field $I\!R[\mathrm{x},\mathrm{y}]$. On the other hand, in Figure 5, we observe $\det(H_{13}(x,y)) = L_{12}(x,y) = x^{12}+12x^{10}y+54x^8y^2+112x^6y^3+105x^4y^4+36x^2y^5+2y^6$ and, by a command of the CAS, we can write $\det(H_{13}(x,y)) = L_{12}(x,y) = (x^4+4x^2y+2y^2)(x^8+8x^6y+20x^4y^2+16x^2y^3+y^4)$ $= L_4(x,y)\cdot L_8(x,y)$, however $L_{12}(x,y)$ is not divisible by $L_2(x,y)$ and 12 is not divisible by 8.

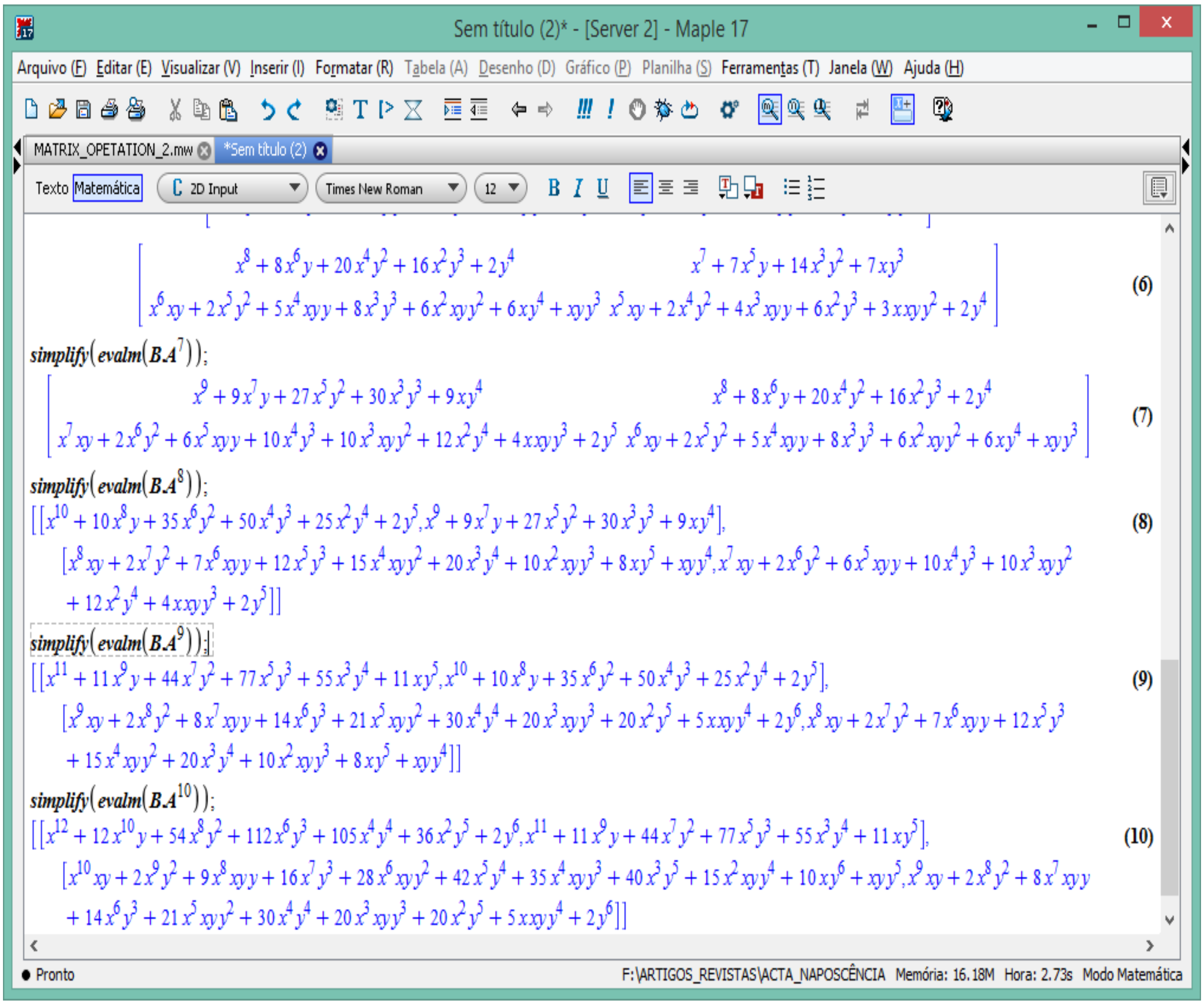

Figura 4. With the *CAS Maple* we can verify the algebraic behavior of the product of the matrices

In addition, with the use of software, we can also determine the factorization and, therefore, the decomposition of irreductible factors of the polynomial terms over the ring $Z[x,y]$. We observe in the list below that, unlike the case of Fibonacci, we will have elements of prime index that admit a factorization in irreducible factors ($L_5(x,y)$, $L_{11}(x,y)$, $L_{17}(x,y)$).

$$\det(H_6(x,y)) = L_5(x,y) = x^5 + 5x^3y + 5xy^2 = x(x^4 + 5x^2y + 5y^2) \text{ (reductible over Z[x,y],p=5)}$$

$$\det(H_{12}(x,y)) = L_{11}(x,y) = x^{11} + 11x^9y + 44x^7y^2 + 77x^5y^3 + 55x^3y^4 + 11xy^5 = L_1(x,y)(x^{10} + 11x^8y + 44x^6y^2 + 77x^4y^3 + 55x^2y^4 + 11y^5) \text{ (reductible over Z[x,y],p=11)}$$

$$\det(H_{13}(x,y)) = L_{12}(x,y) = (x^4 + 4x^2y + 2y^2)(x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + y^4) = L_4(x,y)(x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + y^4) \therefore L_4(x,y) \backslash L_{12}(x,y), \text{is reductible over Z[x,y]}$$

$$\det(H_{15}(x,y)) = L_{14}(x,y) = x^{14} + 14x^{12}y + 77x^{10}y^2 + 210x^8y^3 + 294x^6y^4 + 196x^4y^5 + 49x^2y^6 + 2y^7 = (x^2 + 2y)(x^{12} + 12x^{10}y + 53x^8y^2 + 104x^6y^3 + 86x^4y^4 + 24x^2y^5 + y^6) = L_2(x,y)(x^{12} + 12x^{10}y + 53x^8y^2 + 104x^6y^3 + 86x^4y^4 + 24x^2y^5 + y^6) \text{ is reductible over Z[x,y]}$$

$\det(H_{16}(x,y)) = L_{15}(x,y) = x^{15} + 15x^{13}y + 90x^{11}y^2 + 275x^9y^3 + 450x^7y^4 + 378x^5y^5 + 140x^3y^6 + 15xy^7$
$= x(x^2+3y)(x^4+5x^2y+5y^2)(x^8+7x^6y+14x^4y^2+8x^2y^3+y^4) = L_1(x,y)(x^2+3y)(x^4+5x^2y+5y^2)$
$(x^8+7x^6y+14x^4y^2+8x^2y^3+y^4)$ is reductible over Z[x,y]

$\det(H_{17}(x,y)) = L_{16}(x,y) = x^{16} + 16x^{14}y + 104x^{12}y^2 + 352x^{10}y^3 + 660x^8y^4 + 672x^6y^5 + 336x^4y^6$
$+64x^2y^7 + 2y^8$ (is irreductible over Z[x,y])
$\det(H_{18}(x,y)) = L_{17}(x,y) = x^{17} + 17x^{15}y + 119x^{13}y^2 + 442x^{11}y^3 + 935x^9y^4 + 1122x^7y^5 + 714x^5y^6$
$+204x^3y^7 + 17xy^8 = x(x^{16} + 17x^{14}y + 119x^{12}y^2 + 442x^{10}y^3 + 935x^8y^4 + 1122x^6y^5 + 714x^4y^6 + 204x^2y^7$
$+17y^8)$ (reductible over Z[x,y],p=17)

$\det(H_{19}(x,y)) = L_{18}(x,y) = x^{18} + 18x^{16}y + 135x^{14}y^2 + 546x^{12}y^3 + 1287x^{10}y^4 + 1782x^8y^5 +$
$1386x^6y^6 + 540x^4y^7 + 81x^2y^8 + 2y^9 = (x^2+2y)(x^4+4x^2y+y^2)(x^{12}+12x^{10}y+54x^8y^2+$
$112x^6y^3 + 105x^4y^4 + 36x^2y^5 + y^6) = L_2(x,y)(x^4+4x^2y+y^2)(x^{12}+12x^{10}y+54x^8y^2+112x^6y^3$
$+105x^4y^4 + 36x^2y^5 + y^6)$ is reductible over Z[x,y]

$\det(H_{20}(x,y)) = L_{19}(x,y) = x^{19} + 19x^{17}y + 152x^{15}y^2 + 665x^{13}y^3 + 1729x^{11}y^4 + 2717x^9y^5 + 2508x^7y^6$
$+1254x^5y^7 + 285x^3y^8 + 19xy^9 = x\cdot(x^{18} + 19x^{16}y + 152x^{14}y^2 + 665x^{12}y^3 + 1729x^{10}y^4 + 2717x^9y^5 +$
$2508x^6y^6 + 1254x^4y^7 + 285x^2y^8 + 19y^9) = L_1(x,y)\cdot(x^{18} + 19x^{16}y + 152x^{14}y^2 + 665x^{12}y^3 + 1729x^{10}y^4 +$
$2717x^9y^5 + 2508x^6y^6 + 1254x^4y^7 + 285x^2y^8 + 19y^9)$ $\therefore L_1(x,y)\backslash L_{19}(x,y)$ is reductible over Z[x,y]

Now, from the list of decomposition into irreducible factors of the elements of the bivariate polynomial sequence, we can understand that: $L_1(x,y)\backslash L_3(x,y)$ and $L_3(x,y)$ is not divisible by $L_2(x,y) = x^2+2y$. $L_8(x,y) = x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + 2y^4$ is a irreductible polynomial and is not divisible by $L_2(x,y) = x^2+2y$ or $L_4(x,y) = x^4+4x^2y+2y^2$. In the same manner, the bivariate polynomial $L_{12}(x,y) = x^{12} + 12x^{10}y + 54x^8y^2 + 112x^6y^3 + 105x^4y^4 + 36x^2y^5 + 2y^6$ is reducitible over the ring $IR[\mathrm{x,y}]$, since we obtained $\det(H_{13}(x,y)) = L_{12}(x,y) = L_4(x,y)(x^8+8x^6y+20x^4y^2+16x^2y^3+y^4)$, that is $L_4(x,y)\backslash L_{12}(x,y)$, while $L_{12}(x,y)$ is not divisible by $L_3(x,y)$ or $L_6(x,y)$. On the other hand, we observe $L_3(x,y)\backslash L_{15}(x,y)$ and, however, $L_{15}(x,y)$ is not divisible by $L_5(x,y)$. Moreover, we can conclude that $L_{16}(x,y)$ is not divisible by the following elements $L_2(x,y)$, $L_4(x,y)$ and $L_8(x,y)$. And, the only division propertie is $L_1(x,y)\backslash L_{16}(x,y)$.

According to the result indicated by the software, $L_{17}(x,y)$ is irreducitible polynomial over the ring $IR[\mathrm{x,y}]$. In addition, we further determined that $L_{19}(x,y)$, that despite having a prime subscript, have the element $L_3(x,y)$ as a irreducitible factor. Finally, we conclude that $L_{20}(x,y)$ is not divisible by $L_2(x,y)$, $L_5(x,y)$ or $L_{10}(x,y)$. In Figure 5, we visualize some command employed en virtue to determine it´s decomposition over the ring $Z[\mathrm{x,y}]$.

Now, in addition to an extensive set of algebraic expressions provided by software that indicate the decomposition of polynomial functions into two variables, we can draw some conclusions regarding the character of divisibility and factorization of Lucas' polynomial functions. For example, in Figure 5, on the left side, we observe that $\det(H_9(x,y)) = L_9(x,y) = x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + 2y^4$ is irreductible, since the software when using the factor command ***factor[ ]***, produces the same algebraic expression. But, in the same figure, on the right side, we visualize that the element $\det(H_{13}(x,y)) = L_{12}(x,y) = (x^4 + 4x^2y + 2y^2)(x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + y^4) = L_4(x,y)\cdot(x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + y^4) = L_4(x,y)\cdot L_8(x,y)$ has two components as irreductible polynomials. So, like we have mentioned, we know $L_4(x,y)\backslash L_{12}(x,y)$, however $L_{12}(x,y)$ is not divisible by $L_2(x,y)$.

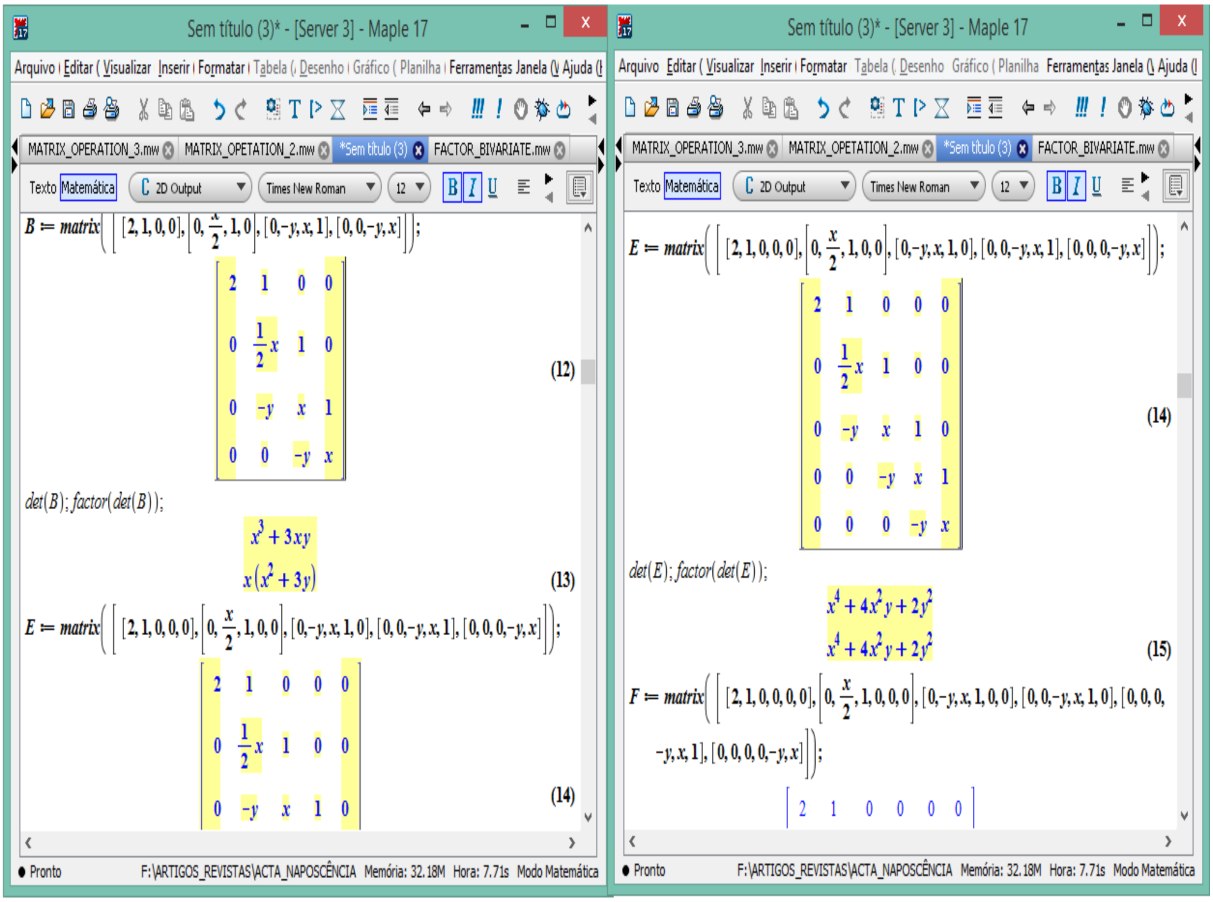

Figura 5. We obtain the decomposition of the irreducible factors of the elements of the set of polynomials with CAS Maple

In the next section, we will address an explicit formula for the polynomial terms present in both sequences. Yet, we will enunciate some properties involving the Greatest Common Divisor of polynomial functions into two variables, now with the introduction of an imaginary unit $i^2 = -1$. In this way we can compare the class of the BFP (definition 4) with the class of BCFP (definition 6). We will find the regularity and invariance of several properties indicated in the theorems discussed here and, conversely, the same regularity cannot be observed in the class of BLP (definition 5) and the BCLP (definition 7). In fact, we showed that $F_m(x,y)\backslash F_n(x,y) \Leftrightarrow \mathrm{m}\backslash\mathrm{n}$. But, with the software, we found several exemples that are counterexamples.

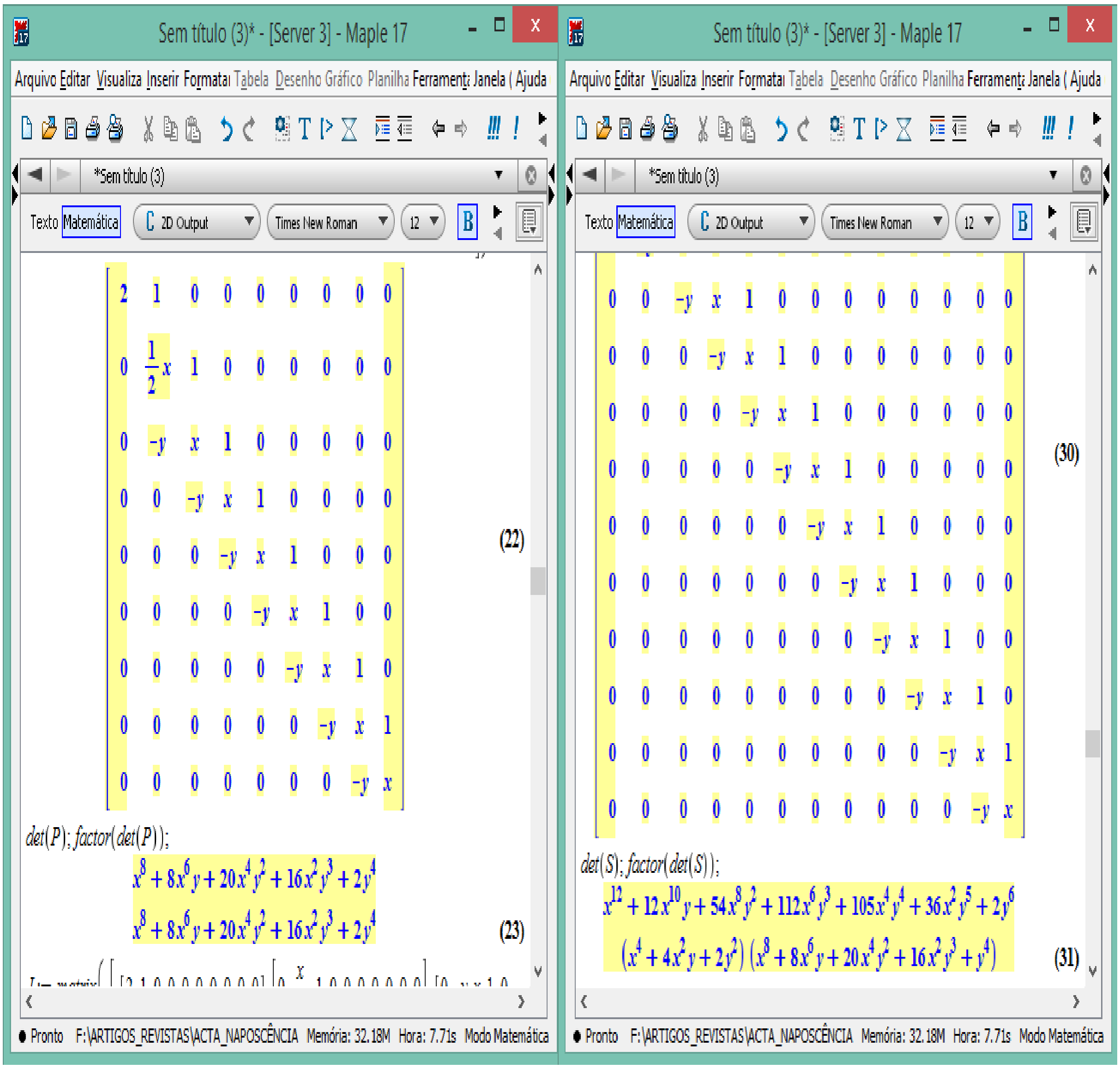

Figura 6. We obtain the decomposition of the irreducible factors of the elements of the set of polynomials with CAS Maple

4. Some properties of the Bivariate Complex Lucas Polynomial Sequence with the Maple´s help.

In the previous section, we discuss some properties of bivariate polynomials in two real variables. We now turn to the study of a special class of Bivariate Complex Fibonacci and Lucas Polynomials, originating from the introduction of an imaginary unit ‘i’ and inherit a tendency of the works interested in the process of complexification of the Fibonacci model (Iakin, 1977; King, 1968; Scott, 1968; Waddill & Sacks, 1967). With this, we can further discuss the process of complexing said recursive sequence. Before, however, we recall the definition presented in the predecessor sections en virtue to present our first theorem.

Theorem 2: For $n > 0$ we have $F_n(x,y) = \dfrac{\left(\alpha^n(x,y) - \beta^n(x,y)\right)}{\alpha(x,y) - \beta(x,y)}$ and $L_n(x,y) = \alpha^n(x,y) + \beta^n(x,y)$.

Proof. We the characteristic equation designated by $t^2 - ix\cdot t - y = 0$. Consequently, we will have the following properties and relations between the roots $\alpha(x,y) = \frac{ix+\sqrt{4y-x^2}}{2}$, $\beta(x,y) = \frac{ix-\sqrt{4y-x^2}}{2}$, $\alpha(x,y)\cdot\beta(x,y) = -y$. Finally, from the recurrence relation, we write: $F_{n+1}(x,y) = ix\cdot F_n(x,y) + y\cdot F_{n-1}(x,y) =$

$$= ix\frac{(\alpha^n(x,y)-\beta^n(x,y))}{\alpha(x,y)-\beta(x,y)} + y\cdot\frac{(\alpha^{n-1}(x,y)-\beta^{n-1}(x,y))}{\alpha(x,y)-\beta(x,y)} = ix\frac{(\alpha^n(x,y)-\beta^n(x,y))}{\alpha(x,y)-\beta(x,y)} + y\cdot\frac{(\frac{\alpha^n(x,y)}{\alpha(x,y)}-\frac{\beta^n(x,y)}{\beta(x,y)})}{\alpha(x,y)-\beta(x,y)}$$

$$= ix\frac{(\alpha^n(x,y)-\beta^n(x,y))}{\alpha(x,y)-\beta(x,y)} + y\cdot\frac{(\alpha^n(x,y)\beta(x,y)-\alpha(x,y)\beta^n(x,y))}{\left(\alpha(x,y)\beta(x,y)\right)\alpha(x,y)-\beta(x,y)} = ix\frac{(\alpha^n(x,y)-\beta^n(x,y))}{\alpha(x,y)-\beta(x,y)} +$$

$$+y\cdot\frac{(\alpha^n(x,y)\beta(x,y)-\alpha(x,y)\beta^n(x,y))}{\left(-y\right)\alpha(x,y)-\beta(x,y)} = \frac{ix\cdot\alpha^n(x,y)-ix\cdot\beta^n(x,y)-\alpha^n(x,y)\beta(x,y)+\alpha(x,y)\beta^n(x,y)}{\alpha(x,y)-\beta(x,y)}$$

$$= \frac{ix\cdot\alpha(x,y)\cdot\alpha^{n-1}(x,y)-ix\cdot\beta(x,y)\cdot\beta^{n-1}(x,y)-\alpha^n(x,y)\beta(x,y)+\alpha(x,y)\beta^n(x,y)}{\alpha(x,y)-\beta(x,y)} =$$

$$= \frac{\alpha(x,y)^{n+1}-y\cdot\alpha^{n-1}(x,y)-\beta(x,y)^{n+1}+y\cdot\beta^{n-1}(x,y)+\mathrm{y}\cdot\alpha^{n-1}(x,y)-y\cdot\beta^{n-1}(x,y)}{\alpha(x,y)-\beta(x,y)} =$$

$= \frac{\alpha(x,y)^{n+1}-\beta(x,y)^{n+1}}{\alpha(x,y)-\beta(x,y)}$. In the similar way, we can determine that $L_n(x,y) = \alpha^n(x,y)+\beta^n(x,y)$.

Corollary 1: For the roots of characteristic equation $t^2 - ix\cdot t - y = 0$ we have the follow identities: $\alpha^n(x,y) = \frac{L_n(x,y)+(\alpha(x,y)-\beta(x,y))F_n(x,y)}{2}, \beta^n(x,y) = \frac{L_n(x,y)-(\alpha(x,y)-\beta(x,y))F_n(x,y)}{2}$.

Proof. From the theorem 1, we take the expression $\left(\alpha(x,y)-\beta(x,y)\right)F_n(x,y) = \alpha^n(x,y)-\beta^n(x,y)$ and $L_n(x,y) = \alpha^n(x,y)+\beta^n(x,y)$. We solve the system $\begin{cases}\alpha^n(x,y)-\beta^n(x,y) = \left(\alpha(x,y)-\beta(x,y)\right)F_n(x,y)\\ \alpha^n(x,y)+\beta^n(x,y) = L_n(x,y)\end{cases}$.

Corollary 2: For $n > 0$ we have $F_n(x,y)\cdot\mathrm{L}_n(x,y) = F_{2n}(x,y)$.

Proof. Bu the previous theorem, we note $F_{2n}(x,y) = \frac{\alpha^{2n}(x,y)-\beta^{2n}(x,y)}{\alpha(x,y)-\beta(x,y)} = \frac{\left(\alpha^n(x,y)-\beta^n(x,y)\right)}{\alpha(x,y)-\beta(x,y)}$

$\cdot\alpha^n(x,y)+\beta^n(x,y)$. Follows the result.

The next results involve some properties related to the divisibility character of the Bivariate Complex Fibonacci Polynomials (BCFP). Thus, with respect to the particular case of the numerical sequence, and especially when we introduce two variables, the properties involving the Greatest Common Divisor (G.C.D) are invariable preserved over the ring $C[x,\mathrm{y}]$. However, with the computational resource, we will see that the same case or pattern does not occur when we deal with the polynomial sequence of Lucas. We can compare with the results established in the preceding sections.

Theorem 2: Let $D_n(x,y)_{n\times n}$ be a tridiagonal matrices. In this case, by the assumption $D_0(x,y)=0$, then we have $\det D_n(x,y)_{n\times n}=F_n(x,y), n\geq 0$. (Asci & Gurel, 2012).

Proof. See Figure 2 (on the left side) and more details can be constated in the Asci & Gurel (2013).

Lemma: Let´s suppose that $mdc(\mathrm{x},\mathrm{y})=1, \mathrm{n}\geq 0$, then $\gcd(\mathrm{y},\mathrm{F}_n(\mathrm{x},\mathrm{y}))=1$.

Proof. More details can be constated in the Asci & Gurel (2013).

Teorema: Suposing that $\gcd(\mathrm{x},\mathrm{y})=1, \mathrm{n}\geq 0$, then $\gcd(\mathrm{F}_n(\mathrm{x},\mathrm{y}),\mathrm{F}_{n+1}(\mathrm{x},\mathrm{y}))=1$.

Proof. More details can be constated in the Asci & Gurel (2013).

Teorema : For $m\geq 2$, we have $m,n\geq 1$ $\mathrm{F}_m(\mathrm{x},\mathrm{y})\setminus \mathrm{F}_n(\mathrm{x},\mathrm{y})\Leftrightarrow \mathrm{m}\setminus \mathrm{n}$.

Proof. More details can be constated in the Asci & Gurel (2013).

Corollary: Suposing that $\gcd(\mathrm{x},\mathrm{y})=1$ e $m,n\geq 1$ then $\gcd(\mathrm{F}_m(\mathrm{x},\mathrm{y}),\mathrm{F}_n(\mathrm{x},\mathrm{y}))=F_{\gcd(m,n)}(x,y)$.

Proof. More details can be constated in the Asci & Gurel (2013).

However, when we seek to inspect similar properties in the Lucas sequence, from some particular cases anticipated by the software, we may notice that several properties are lost and not verified. In fact, we list some cases below relatively to the BCLP.

$\det(H_{13}(x,y))=L_{12}(x,y)=x^{12}-12x^{10}\mathrm{y}+54x^8y^2-112x^6y^3+105x^4y^4-36x^2y^5+2y^6=(x^4-4x^2y+2y^2)$ $(x^8-8x^6y+20x^4y^2-16x^2y^3+y^4)=\mathrm{L}_4(\mathrm{x,y})\cdot(x^8-8x^6y+20x^4y^2-16x^2y^3+y^4)$ is reductible over Z[x,y]

$\det(H_{14}(x,y))=L_{13}(x,y)=x^{13}i-13x^{11}yi+65x^9y^2i-156x^7y^3i+182x^5y^4i-91x^3y^5i+13xy^6i=$ $\mathrm{ix}\cdot(x^{12}-13x^{10}y+65x^8y^2-156x^6y^3+182x^4y^4-91x^2y^5+13y^6)$ is reductible over Z[x,y]

$\det(H_{16}(x,y))=L_{15}(x,y)=-x^{15}i+15x^{13}yi-90x^{11}y^2i+275x^9y^3i-450x^7y^4i+378x^5y^5i-140x^3y^6i$ $+15xy^7i=-ix\cdot(x^2-3y)\cdot(x^8-7x^6y+14x^4y^2-8x^2y^3+y^4)\cdot(x^4-5x^2y+5y^2)$
is reductible over Z[x,y]

$\det(H_{17}(x,y))=L_{16}(x,y)=x^{16}-16x^{14}y+104x^{12}y^2-352x^{10}y^3+660x^8y^4-672x^6y^5+336x^4y^6-64x^2y^7+2y^8$
is irreductible over Z[x,y]

$\det(H_{18}(x,y))=L_{17}(x,y)=x^{17}i-17x^{15}yi+119x^{13}y^2i-442x^{11}y^3i+935x^9y^4i-1122x^7y^5i+714x^5y^6i$ $-204x^3y^7i+17xy^8i=ix\cdot(x^{16}-17x^{14}y+119x^{12}y^2-442x^{10}y^3+935x^8y^4-1122x^6y^5+714x^4y^6-$ $204x^2y^7+17y^8)$ is reductible over Z[x,y]

$\det(H_{19}(x,y)) = L_{18}(x,y) = -x^{18} + 18x^{16}y - 135x^{14}y^2 + 546x^{12}y^3 - 1287x^{10}y^4 + 1782x^8y^5 - 1386x^6y^6 + 540x^4y^7$
$-81x^2y^8 + 2y^9 = -(x^2-2y)\cdot(x^4-4x^2y+y^2)\cdot(x^{12}-12x^{10}y+54x^8y^2-112x^6y^3+105x^4y^4-36x^2y^5+y^6) =$
$= L_2(x,y)\cdot L_4(x,y)\cdot(x^{12}-12x^{10}y+54x^8y^2-112x^6y^3+105x^4y^4-36x^2y^5+y^6)$
is reductible over Z[x,y],p=18

$\det(H_{20}(x,y)) = L_{19}(x,y) = -x^{19}i + 19x^{17}yi - 152x^{15}y^2i + 665x^{13}y^3i - 1729x^{11}y^4i + 2717x^9y^5i$
$-2508x^7y^6i + 1254x^5y^7i - 285x^3y^8i + 19xy^9i = -ix\cdot(x^{18} - 19x^{16}y + 152x^{14}y^2 - 665x^{12}y^3 + 1729x^{10}y^4$
$-2717x^8y^5 + 2508x^6y^6 - 1254x^4y^7 + 285x^2y^8 - 19y^9)$(is reductible over Z[x,y],p=19)

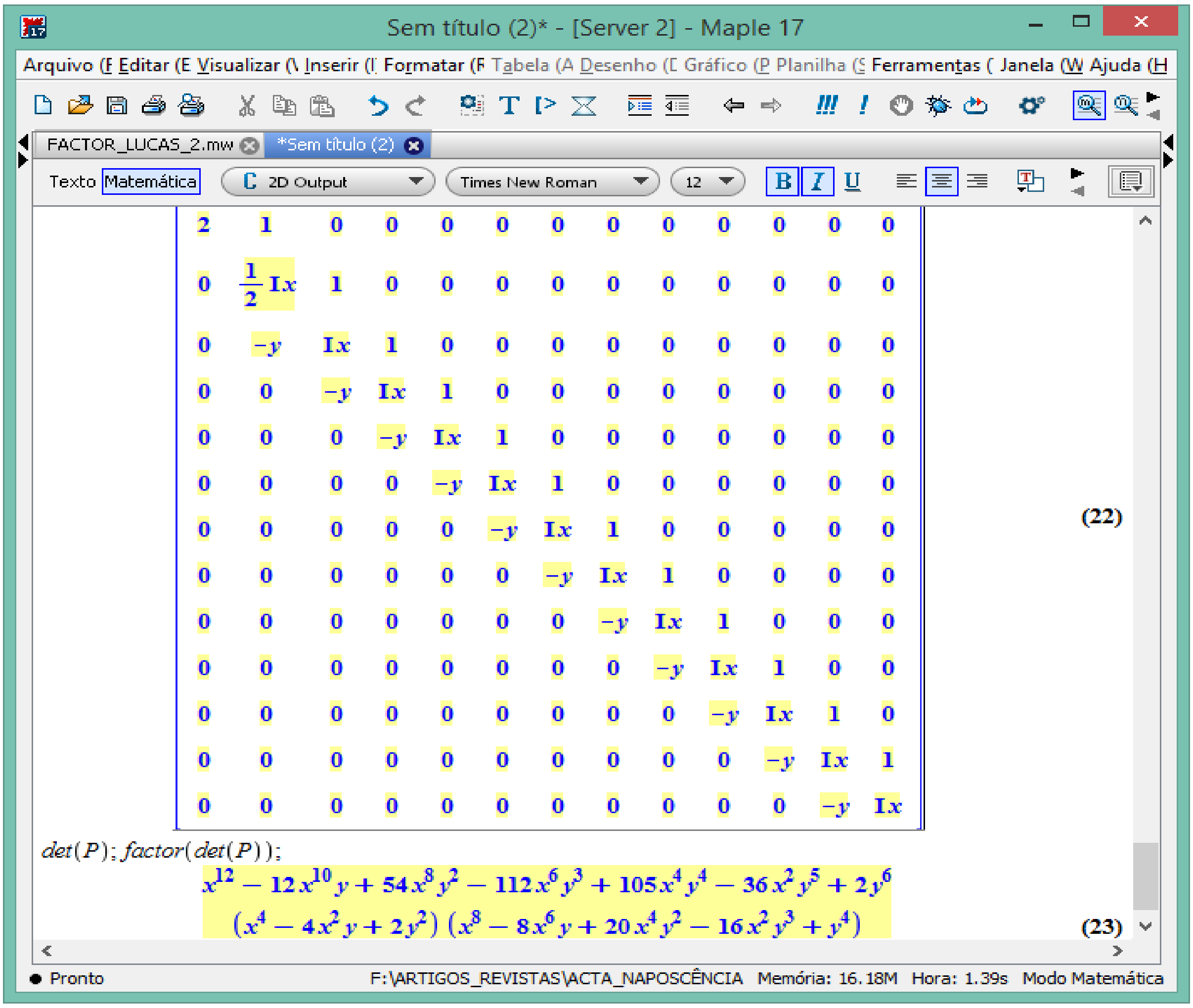

Figura 7. Factoring of terms of the bivariate sequence of Lucas with the CAS Maple (produced by the authors).

Now, in summary, we show that $L_{13}(x,y)$ is reductible over □$[x,y]$. $L_{14}(x,y)$ is divisible by $L_2(x,y) = \text{ix}$, but is not divisible for $L_7(x,y)$. $L_{16}(x,y)$ is divisible by $L_2(x,y) = \text{ix}$, but is not divisible by $L_4(x,y) = -x^3i + 2xyi$.

On the other hand, $L_{16}(x,y) = x^{16} - 16x^{14}y + 104x^{12}y^2 - 352x^{10}y^3 + 660x^8y^4 - 672x^6y^5 + 336x^4y^6 - 64x^2y^7 + 2y^8$ is a irreductible polynomial, while $L_{18}(x,y)$ is not divisible by $L_3(x,y), L_9(x,y)$ and is divisible by $L_2(x,y) = -x^2 + 2y$ and $L_4(x,y) = x^4 - 4x^2y + y^2$. Finally, $L_{19}(x,y)$ has two irreducible factors and none of them belongs to Lucas Polynomial´ sequence. Thus, similar to the case of the Lucas´ sequence we saw in the last section, when we deal with the introduction of the imaginary unit, we realize that the same properties that we study with the Fibonacci sequence are lost.

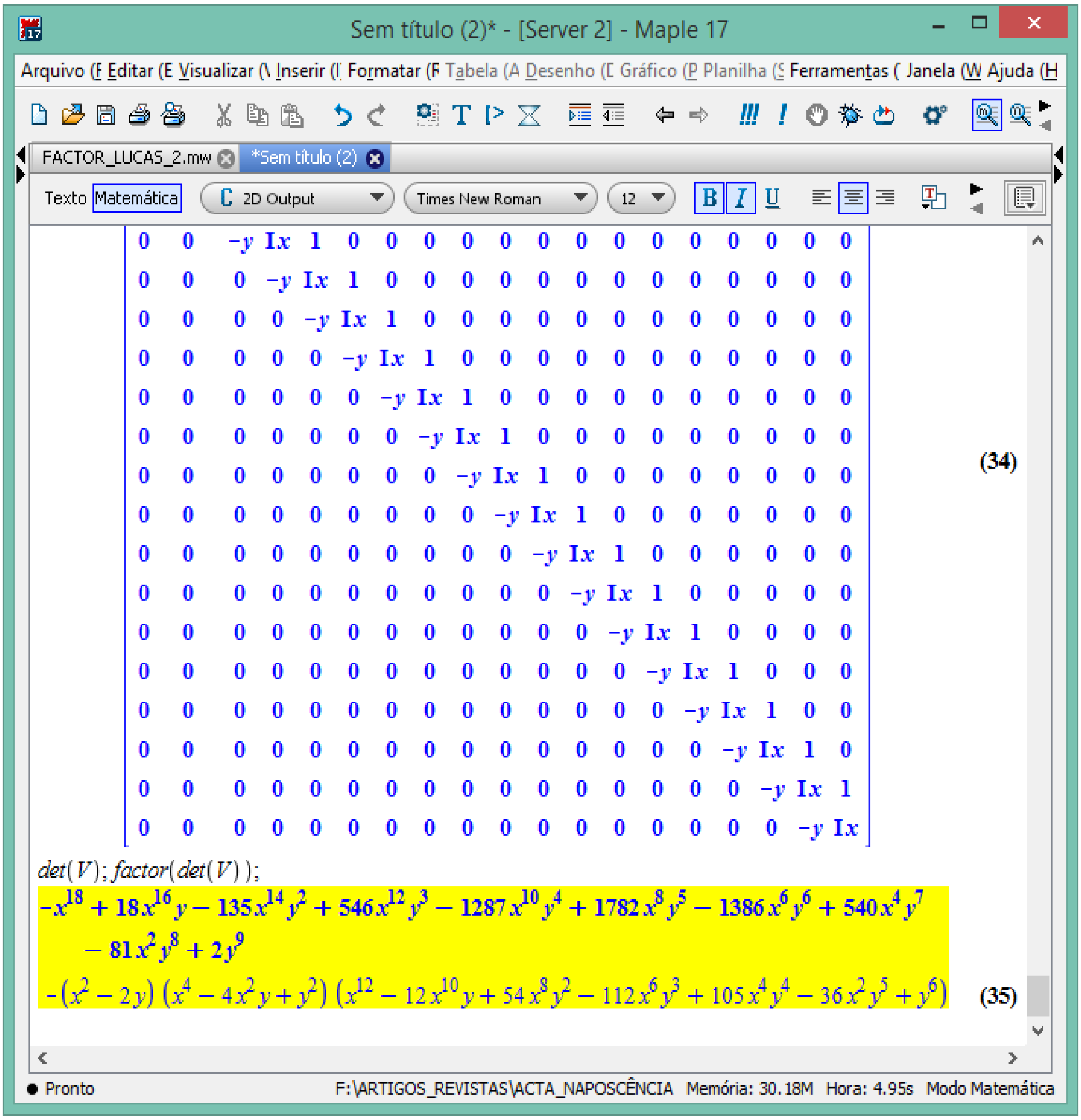

Figura 8. Determination of an element of nineteen order in the BLP $L_{19}(x,y)$. (produced by the authors).

4. An historical investigation with the Maple´s help

In some of our works (Alves, 2017; 2016a; 2016b; 2016c; 2015) that seek to emphasize a perspective that points out the importance of the History of Mathematics, with a view to an understanding of the process of emergence, evolution and systematization of ideas, models, abstract properties, theorems and, above all, formal definitions. We have taken a position that Mathematics envolves as a process of debugging, synthesizing and testing the adequacy and robustness of certain formal mathematical definitions. Thus, we assume that for an evolution of the evolutionary process in Mathematics, an understanding about the process that contributes to such demarcation is essential.

On the other hand, certain definitions determine a charged notational representation system that sometimes prevents an understanding and the establishment of necessary conceptual relations in order to identify the invariant elements in each case. With this function, CAS Maple allowed us to explore several particular cases, verifying and proving properties originating from an inductive model or reasoning. In addition, CAS Maple makes it possible to perform calculations that are impractical when we disregard the technology.

Regarding the Maple´s use, we highlight the elements: (i) The software enables verifications of particular cases and properties related to the Bivariate Fibonacci Polynomials and the Bivariante Lucas Polynomials; (ii) The software allows the verification properties provided by classical theorems related to the Bivariate (Complex) Fibonacci Polynomials and Bivariante (Complex) Lucas Polynomials, especially the most recently discussed in the literature; (iii) The software allows the description of a lot of special particular cases conditioned by newly formulated mathematical definitions; (iv) The software enables verification of properties related to a larger set of integer subscripts indicated in the scientific articles; (v) the software enables verification of a large number of individual cases in order to test mathematical conjectures about the BFP and BLP; (vi) the software allows the correction of mathematical formulas in order to provide a precise description.

Before concluding the current section, we bring a last formal definition that expresses the current character of the discussion of the subject addressed to the course of this work. In the last definion we constate the Fibonacci` model described by the complex variable $z = x + iy$.

Definition 4: Let a Generalized Polynomial $\{F_n(a,z)\}_{n=0}^{\infty}$ in the variables 'a' and 'z', is designed for the recurrence relation $F_n(a,z) = \mathrm{a} \cdot \mathrm{z} \cdot F_{n-1}(a,z) + \mathrm{a}^2 \cdot F_{n-2}(a,z), \mathrm{n} \geq 2$, with initial conditions $F_0(a,z) = 0, F_1(a,z) = 1$. (TASKÖPRÜ & ALTINTAS, 2015).

In the particular case for the value $a = 1$, $f_2(z) = z$ $f_3(z) = z^2 + 1$, $f_4(z) = z^3 + 2z = z(z^2 + 2)$, $f_5(z) = z^4 + 3z^3 + 1$, $f_6(z) = z^5 + 3z^4 + z^3 + 3z = z(z^2+3)(z^2+1)$, $f_7(z) = z^6 + 5z^4 + 6z^2 + 1$, $f_8(z) = z^7 + 6z^5 + 10z^3 + 4z = z(z^2+2)(z^4+4z^2+2)$, $f_9(z) = z^8 + 7z^6 + 15z^4 + 10z^2 + 1 = (z^2+1)(z^6+6z^4+9z^2+1)$, $f_{10}(z) = z^9 + 8z^7 + 21z^5 + 20z^3 + 5z = z(z^4+3z^2+1)(z^4+5z^2+5)$, $f_{11}(z) = z^{10} + 9z^8 + 28z^6 + 35z^4 + 15z^2 + 1$, $f_{12}(z) = z^{11} + 10z^9 + 36z^7 + 56z^5 + 35z^3 + 6z = z(z^2+3)(z^2+2)(z^2+1)(z^4+4z^2+1)$, $f_{13}(z) = z^{12} + 11z^{10} + 45z^8 + 84z^6 + 70z^4 + 21z^2 + 1$, $f_{14}(z) = z^{13} + 12z^{11} + 55z^9 + 120z^7 + 126z^5 + 56z^3 + 7z = z(z^6+5z^4+6z^2+1)(z^6+7z^4+14z^2+7)$, $f_{15}(z) = z^{14} + 13z^{12} + 66z^{10} + 165z^8 + 210z^6 + 126z^4 + 28z^2 + 1 = (z^2+1)(z^4+3z^2+1)(z^8+9z^6+26z^4+24z^2+1)$, $f_{16}(z) = z^{15} + 14z^{13} + 78z^{11} + 220z^9 + 330z^7 + 252z^5 + 84z^3 + 8z = z(z^2+2)(z^4+4z^2+2)(z^8+8z^6+20z^4+16z^2+2)$, $f_{17}(z) = z^{16} + 15z^{14} + 91z^{12} + 286z^{10} + 495z^8 + 462z^6 + 210z^4 + 36z^2 + 1$, $f_{18}(z) = z^{17} + 16z^{15} + 105z^{13} + 364z^{11} + 715z^9 + 792z^7 + 462z^5 + 120z^3 + 9z$ $z(z^2+3)(z^2+1)(z^6+6z^4+9z^2+1)(z^6+6z^4+9z^2+3)$, $f_{19}(z) = z^{18} + 17z^{16} + 120z^{14} + 455z^{12} + 1001z^{10} + 1287z^8 + 924z^6 + 330z^4 + 45z^2 + 1$, $f_{20}(z) = z^{19} + 18z^{17} + 136z^{15} + 560z^{13} + 1365z^{11} + 2002z^9 + 1716z^7 + 792z^5 + 165z^3 + 10 = z(z^2+2)(z^4+3z^2+1)(z^4+5z^2+5)(z^8+8z^6+19z^4+12z^2+1)$ .etc.

Again, we can observe the extensive list above of polynomials in the complex variable. The software allows us to list and perform the decomposition (factorization) of high order elements. From an appreciation in the list above, we can identify some cases of irreducible polynomials when we have the condition $F_n(\mathrm{z})$ is ireductible, $n = p$ is prime. On the other hand, from the list and comparing it with the theorems of the previous section, we can conjecture that the divisibility properties can be derived also in the case of the complex variable. Now, from the above relationships, we conjecture the following results

Conjecture 1: For $m \geq 2$, we have $m, n \geq 1$ $\mathrm{F}_m(\mathrm{z}) \setminus \mathrm{F}_n(\mathrm{z}) \Leftrightarrow \mathrm{m} \setminus \mathrm{n}$.

Conjetucture 2: For $m, n \geq 1$ then $\gcd(\mathrm{F}_m(\mathrm{z}), \mathrm{F}_n(\mathrm{z})) = F_{\gcd(m,n)}(\mathrm{z})$.

Conjetucture 3: $F_n(\mathrm{z})$ is irreductible, when $n = p$ is a prime.

Certainly, the previous conjectures can be verified in some particular cases in order to predict their chances of success or the existence of possible counterexamples. This type of procedure or concern is recurrent and standard in Mathematics.

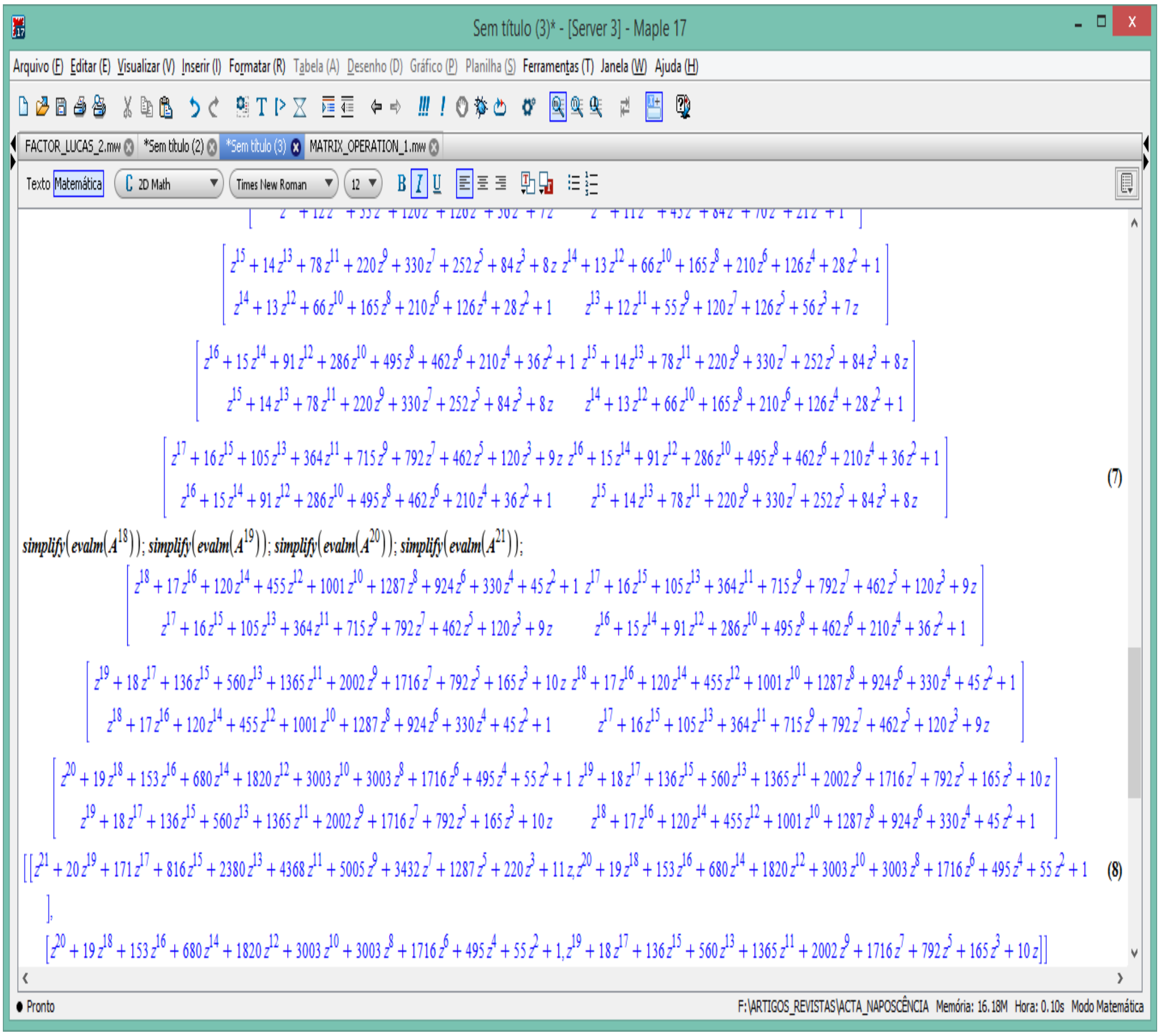

Figura 9. The software allows the determination and decomposition of Fiubonacci polynomial functions in the complex variable. (produced by the authors).

To conclude, in the table below we bring a comparative and conceptual picture of the properties we have discussed so far, with emphasis on the Lucas sequence model. In a brief and simplified way, we invite the reader to appreciate the existing framework and conceptual relationships closely related to the numerical sequences we discussed in the introductory section. We recall the basic properties of values $\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}$ the roots of the equation $x^2-x-1=0$ (Koshy, 2007; 2011; Honsberger, 1985; Stakhov, 2005; 2009; Tattersall, 2005).

Table 1: Comparative table of properties related to the Lucas and Fibonacci model.

Arithmetic properties	Fibonacci´s model	Lucas´ model
Let $k,n \in IN$, with k odd, then $f_n \setminus f_{k\cdot n}$ and $L_n \setminus L_{k\cdot n}$.	$\mathrm{F}_m(\mathrm{x},\mathrm{y}) \setminus \mathrm{F}_n(\mathrm{x},\mathrm{y}) \Leftrightarrow \mathrm{m} \setminus \mathrm{n}$ (Web & Parberry, 1969).	Corresponding property is not true. $L_4(x,y)=x^4+4x^2y+2y^2$ and $L_{16}(x,y)$ are irreductible. $L_{19}(x,y)$ is reductible over $Z[x,y]$.
Let $k,n \in IN$, with k even, then $f_n \setminus f_{k\cdot n}$ and $L_n \setminus L_{k\cdot n}$.	$\mathrm{F}_m(\mathrm{x},\mathrm{y}) \setminus \mathrm{F}_n(\mathrm{x},\mathrm{y}) \Leftrightarrow \mathrm{m} \setminus \mathrm{n}$ (Web & Parberry, 1969). ${F_n}^2 \setminus F_{n\cdot m} \Leftrightarrow F_n \setminus \mathrm{m}$ (Matijasevi˘c, 1970).	Corresponding property is not true. $L_{12}(x,y)=L_4(x,y)\cdot L_8(x,y)$.
Let $\mathrm{m},n \in IN$ and let $d=\gcd(m,n)$, then $\gcd(f_m,f_n)=f_d$.	$\gcd(\mathrm{F}_m(\mathrm{x},\mathrm{y}),\mathrm{F}_n(\mathrm{x},\mathrm{y}))=F_{\gcd(m,n)}(x,y)$	Corresponding property is not true. $\gcd(\mathrm{L}_m(\mathrm{x},\mathrm{y}),\mathrm{L}_n(\mathrm{x},\mathrm{y})) \neq L_{\gcd(m,n)}(x,y)$ $\gcd(\mathrm{L}_{12}(\mathrm{x},\mathrm{y}),\mathrm{L}_6(\mathrm{x},\mathrm{y}))=1 \neq \mathrm{L}_6(\mathrm{x},\mathrm{y})$ BLP
Let $\mathrm{m},n \in IN$ and let $d=\gcd(m,n)$. If the numbers $\frac{m}{d},\frac{n}{d}$ are both odd, then $\gcd(L_m,L_n)=L_d$.	$\gcd(\mathrm{F}_m(\mathrm{x},\mathrm{y}),\mathrm{F}_n(\mathrm{x},\mathrm{y}))=F_{\gcd(m,n)}(x,y)$ With the imaginary unit $i^2=-1$. BCFP	Corresponding property is not true. $\gcd(\mathrm{L}_m(\mathrm{x},\mathrm{y}),\mathrm{L}_n(\mathrm{x},\mathrm{y})) \neq L_{\gcd(m,n)}(x,y)$ and $L_8(x,y)=x^8-8x^6y+20x^4y^2-16x^2y^3+2y^4$ Is irreducitible. $\gcd(\mathrm{L}_8(\mathrm{x},\mathrm{y}),\mathrm{L}_4(\mathrm{x},\mathrm{y})) \neq L_4(x,y)$ (BCLP)
$f_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, $L_n=\alpha^n+\beta^n$	$F_n(x,y)=\frac{\left(\alpha^n(x,y)-\beta^n(x,y)\right)}{\alpha(x,y)-\beta(x,y)}$ With the imaginary unit $i^2=-1$.	$L_n(x,y)=\alpha^n(x,y)+\beta^n(x,y)$ With the imaginary unit $i^2=-1$. (BCPL)
$f_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, $L_n=\alpha^n+\beta^n$	$F_n(\mathrm{a},\mathrm{z})=\frac{\left(\alpha^n(\mathrm{a},\mathrm{z})-\beta^n(\mathrm{a},\mathrm{z})\right)}{\alpha(\mathrm{a},\mathrm{z})-\beta(\mathrm{a},\mathrm{z})}$ Where ‘z’ is a complex variable (Tasköprü & Altintas, 2015)	$L_n(\mathrm{a},\mathrm{z})=\alpha^n(\mathrm{a},\mathrm{z})+\beta^n(\mathrm{a},\mathrm{z})$ Where ‘z’ is a complex variable (Tasköprü & Altintas, 2015)
f_p or L_p, where $p \in IN$ is prime. $f_{19}=113\cdot 37$	$F_p(x,y)$ and ‘p’ is prime. Then $F_p(x,y)$ is irreductible over the ring $\mathrm{Z}[x,y]$. (Hoggatt & Long, 1974)	Corresponding property is not true. $L_1(x,y) \setminus L_{13}(x,y)$ $L_1(x,y) \setminus L_{19}(x,y)$

(Web & Parberry, 1969).	Conjecture: $F_p(z)$ is irreductible over the ring $C[x,y]$	$L_{12}(x,y)$ is not divisible by $L_2(x,y)$ and $L_3(x,y)$.

3. Conclusion

In this work we present a set of mathematical definitions related to a generalized sequence model that allow an understanding of the evolutionary epistemological and mathematical process of a second order recurrent sequence, originally systematically studied by the French mathematician François Édouard Anatole Lucas (1842 – 1891) and, another formulated by Leornardo Pisano, in 1202. In addition, the problem of factorization of numbers present in both sequences $\{f_n\}_{n\in IN}$ and $\{L_n\}_{n\in IN}$ consists of a problem of interest studied in the field of Pure Mathematics and Mathematics Applied for decades. Thus, we can find some particular cases in the specialized literature that certain expected divisibility properties do not occur, for both numerical sequences (Onphaeng & Pongsriiam, 2014; Pongsriiam, 2014). And in many cases, only with the evolution of technology has it become possible to test and verify the numerical behavior of a sequence, for increasing indices (see Figure 3).

Moreover, given these elements and others that we seek to discuss in all sections, mainly some elements with respect to an evolutionary epistemological trajectory and, especially, an historical perspective (see definitions 1, 2, 3, 4, 5, 6 and 7). In this way, it may raise an understanding about the continued ans unstopable progress in Mathematics and some elements, which can contribute to an investigation about these sequences, which is customarily discussed in the academic environment, however in relation to their formal mathematical value.

In view of the use of Computational Algebraic System Maple, we have explored particular situations enabling a heuristic thought and not completely accurate and precise with respect to certain mathematical results (see figures 4, 5, 6, 7 and 8). Such situations involved checking of algebraic properties extracted from numerical, algebraic and combinatorial formulations of the Fibonacci's Polynomial model and the Lucas´Polynomials model, for example, discussed by Hoggatt & Bicknell (1973a; 1973b). In addition, with the software we can determine the explicit behavior of several parficular cases in order to determine possible counterexamples for the both models discussed in this work, above all, related to the Lucas´ model, for example $L_{12}(x,y)=L_4(x,y)\cdot L_8(x,y)$.

Another important role is due to the detailed study and epistemological appreciation of the formal mathematical definitions addressed throughout the work (set of seven definitions). We assume a position that Mathematics progresses and it´s progress can be appreciated from the progressive establishment of mathematical definitions (Alves, 2017; 2016a; 2016b. 2016c, 2015), since they constitute the marked elements of the solidity and certainty of mathematical assertions and therefore of theorems discussed here.

Finally, since all the arguments and properties presented are closely related to the Fibonacci´s model, an understanding of it´s mathematical, epistemological, and evolutionary process cannot be disregarded or reduced to a kind of Mathematical History´s textbook approach (Arcavi & Isoda, 2007). Restricted in an eminently playful and fictional discussion of the Fibonacci episode, in view of the problem of the birth of pairs of rabbits.

In this way, we present to the reader elements and information derived from a formal and computational mathematical model, in order to demonstrate a greater, complete and broad understanding of some studies on the subject that require a greater scientific discussion and dissemination, in order to provide an evolutionary understanding about the current research of some subjects in Mathematics. Table 1 should provide a systematic and simplified view of the various properties (related to the BFP, BLP, BCFP and BCLP) discussed throughout the paper.

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Authors

Francisco Regis Vieira Alves, Mathematics Department. Federal Institute of Science and Technology of Ceara - IFCE, Fortaleza, Brazil, e-mail: fregis@ifce.edu.br

Paula Maria Machado Cruz Catarino, Mathematics Deparment, Universidade Trás-os-Montes e Alto Douro – UTAD. Vila Real, Portugal. e-mail: pcatarino23@gmail.com