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ABOUT THE TEACHING - LEARNING CONCEPT OF "FUNCTION" IN GYMNASIUM

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Abstract: This paper briefly describes the process of teaching – learning the concept of "*function*" (or "*map*") in middle school. Thus, there are practical matters of everyday life that lead to the introduction of the notion of function, the definition of this notion, numerous examples and counterexamples, as well as the perspective directions in forming the concept of function in high school. Throughout this paper we use both nems: function and map.

Key words: mathematics, relation, equality, function, correspondence, argument, equations, solutions, groups, fields.

1. Introduction

Teaching the notion of function (map) does not mean saying, dictating or writing on the board the definition and asking the students to reproduce it in a written or oral form the next class. Teaching the notion of function (map), as well as teaching the notions of set and relation, means:

- presenting close-to-real life examples;
- leading the student towards analyzing and comparing;
- organizing and supervising the learning of the notion;
- making the learned notions functional and operating with them in exercises and problem solving.

In this paper we intend to present some methodological aspects of concerning the teaching and learning on the concept of function in the middle school classes from Romania. Thus are presented the most important steps how can one introduce the concept of function by real life examples, how the teacher must to lead the students towards analysing and comparing, how to organize and supervise the learning of the notion, how to make the notion functional and how to operate with functions in problem solving - following the steps listed in [5], [6] or [7].

Of course that teaching of specific mathematical knowledge to a class of students is made according the:

- what the objectives to be achieved in this regard,
- competences that need to be formats students,
- objectives and competences presented in the school curricula or proposed by the teacher -
- the skills and abilities (intellectual and psychomotor) which will form the students,
- the contents of the curricula,
- the level of intellectual development and motivation of those students of learning Mathematics.

No teacher can not to assert that holds the universal recipe, because everything is contextualizes. There are interesting papers, older or newer, concerning the the subject matter. The reader interested in these issues, we recommend the works of the bibliographical list, see, particularly [3], [4], [8], [9] and [10].

What we present here is a model, a point of view, certainly not the only one, nor the best, but in the classes that has been applied has given very good results, students have formed a first good image, the consistent on this concept: function or application.

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2. A model of a lesson and not only that

As in daily life, the concept of function (map) plays an important role not only in Mathematics, but also in the whole Science – see [2].

The elements of set theory will contribute to a large extent to the profound understanding and application of the notion of function (map).

Therefore, in the 7th grade, one begins with two sets and builds a functional relation of dependence between the two sets. Examples leading to this particular relationship will be taken from daily life, either from the previous grades, or from other school objects.

Examples 1: A mobile is moving in straight line with a speed of 60km/h.

1) If we mark with x the time of movement (expressed in hours), how do we calculate the distance (space expressed in km)?

Using the knowledge from Physics, students will answer:

$d=s\cdot t$ (d- distance; s – speed, t – time)

$y=60\cdot x$.

2) What kind of dependence is established between the two quantities: distance and time, respectively between the two sets A and B?

Using the 6th grade knowledge about direct and inverse proportions, students will answer:

the longer the time, the greater the distance;

furthermore, if time grows by a number of time, the distance grows by the same number of time; the results of the table can be expressed by a line of direct proportions:

$$\frac{1}{60}=\frac{2}{120}=\frac{3}{180}=\frac{5}{300}=\frac{10}{600},$$

therefore, time and distance are in directly proportional.

3) Draw a table and then a diagram according to the relation:

$x\in A=\{1,2,3,5,10\}$,

given the second set:

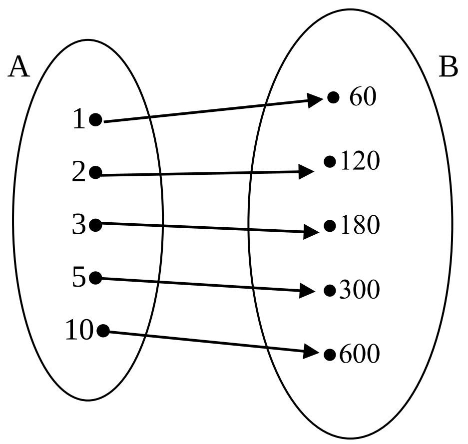
$B=\{y \mid y=60\cdot x,\ x\in A\}$.

After determining the set:

$B=\{60,120,180,300,600\}$,

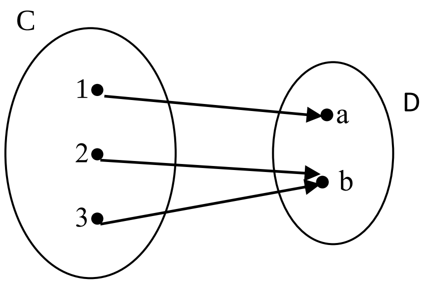
students will write the answers:

movement duration (time) x	1	2	3	5	10
space covered (distance) 60x	60	120	180	300	600

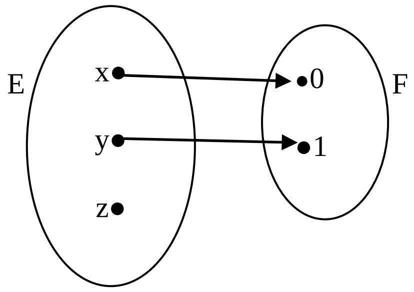


In order to get students to answer the second part of question number 2), the teacher must lead the students to discover the rule by which the sets A and B are in a relation of functional dependence and for this more examples will be given.

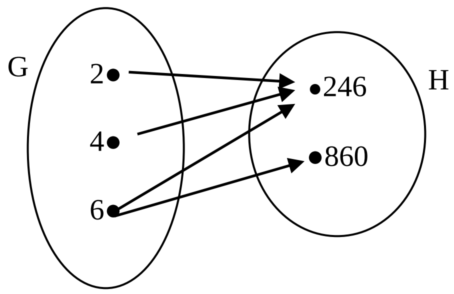
Example 2: The correspondence from the diagram below is a function (map) f : C → D.



Counterexample 3: The correspondence from the diagram below is not a function (map) because no element in F corresponds to the element z∈E.



Counterexample 4: The correspondence from the diagram below is not a function (map) because for element 6∈G there are two corresponding elements (both 246 and 860) from H.



The students will make the following remarks:

- for every element from the set A there is only one correspondent from the set B;
- for every element from the set C there is only one correspondent from the set D;
- for element z from the set E there is no corresponding element in set F;
- for element 6 from the set G there are two correspondent elements in seta H: both 246 and 860.

The teacher will give the following explanations:

- we say that A and B are in a relation of functional dependence;
- we say that E and F, respectively G and H are not in a relation of functional dependence.
- The definition must be short, clear and concise:

> “*We say that two sets A and B are in a relation of functional dependence if, for every element in A there is a correspondent element in B*”.

At the end of the 7th grade, the student must be able to recognize the sets which are in a relation of functional dependence, to provide / build examples of functional dependence and represent them in a table, diagram and formula. So, the emphasis is placed on *sets* and on the *relation* established between them, the word *function (map)* being omitted, as well as its symbols.
In the 8th grade the following objectives are aimed identifying functions (maps) of the type f : **R** → **R**, f(x)=ax+b (a, b∈**R**) and graphic representations of these.

To reach this aim, the following teaching activities will be employed:

- analyzing examples of functional dependence met in other school objects;
- building examples of functional dependence;
- exercise of writing the formula which defines a functional dependence defined on a finite set (in the case of simple formulas);
- analyzing and building examples to illustrate the notions of function (map), diagram, function (map) defined on a finite set, function (map) defined on an infinite set, graphic;
- finding the sets of the values of a function (map) defined on a finite set;
- exercises of graphic representation of functions (maps) defined on finite sets, or defined on **R** with values in **R**, f(x)=ax+b, a, b∈**R**, in a octagonal axis system;
- exercises of determining a function of the type f : **R** → **R**, f(x)=ax+b, whose graphic contains two given points;
- exercises of investigating the collinearity of two or more points, knowing their coordinates.

In most alternative course books for the 8th grade, the notion of function (map) is introduced starting from the usage of the world in daily speech, through frontal catechetic conversation.

Examples 5: 1) Computer breakdown in the ninth grade - to be *function on* average;
2) He has an important *function* in the company;
3) We dress according / *function* to weather ([1]).

Then, one resorts to previously acquired knowledge from other school objects.

Examples 6: 1) In Biology you have learnt about the structure and *functions* of the organism.
2) In Physics you have learnt that a mobile moving with constant speed, straightly and evenly, will cross a space proportional to the time of movement, according to the well-known formula d=s·t.
In this formula, the speed v is a constant quantity and the duration t is variable. The formula sets a relation between duration t and the distance s, namely that the distance depends on the duration of movement. If we write the formula in the form s(t)=v·t, then we express the distance as a *function (map)* of the duration of movement.

Examples 7: The relations in Examples 1 and 2 are functions (maps), and the Counterexamples 3 and 4 are not functions (maps), but just relations established between the elements of the two sets.

Based on the previous knowledge referring to functional dependence and based on the given examples, one may give the definition of the mathematic function (map), with its symbolism, only after emphasizing the three elements involved: *two sets* and *a rule* (law of correspondence or association) which provides for each element in the first set only one corresponding element in the second set.

> *"Given two non-empty sets A and B and a correspondence law f which provides for every x element in A a unique corresponding y element in B, we define a function (map) in A, with values in B and write:*
> $f : A \to B$ or $A \xrightarrow{f} B$".

The specific terminology will be exemplified with one of the functions (maps) from the previous examples (Example 1):

f : {1,2,3,5,10} → {60,120,180,300,600},

where:

f(x)=60x;
A={1,2,3,5,10}: domain of definition;
B={60,120,180,300,600}: co-domain or the set in which the function (map) has values;
x∈{1,2,3,5,10}: arguments;
y∈{60,120,180,300,600}: values or images;
y=f(x): law of correspondence;
Imf=f(A)={60,120,180,300,600}: the set of values or images of the function (map).

Using the same Example 1, students have already learnt the three ways of defining a function (map) (table, diagram and formula). To ensure retention, another example of function (map) will be given and the students will be required to represent it in the three ways. However, teacher will draw attention to the fact that the function (map) is a triplet: (A,B,f).

The Mathematics curriculum for the 8^{th} grade also includes the geometric representation of the function (map): $f : A \to \mathbf{R}$, $f(x)=ax+b$, where A is a finite set or $A=\mathbf{R}$ or A is an interval of real numbers.

If the majority of students are able to make the graphic representation of the functions (maps) such as the above mentioned, only fewer solve Geometry problems using their graphic representations. Therefore, in order for Algebras notions referring to function (map) to become available for application of students at another level (high-school) and in other branches of Mathematics (Geometry, Mathematic analysis) and other school objects (Physics), we will enumerate some of the geometric applications of functions (maps) for the 8^{th} grade, which can be followed by exercises and problems:

1) establishing whether a point belongs or not to the graphic of a function (map);
2) determining functions (maps) of the type $f : \mathbf{R} \to \mathbf{R}$, $f(x)=ax+b$ whose graphic contains two given points;
3) checking the collinearity of three or more given points;
4) determining points on the graphic in certain conditions (for example, they have one coordinate expressed as a function (map) of the other coordinate);
5) determining intersection points of a first degree function’s (map’s) graphic with the coordinate axis;
6) calculating the area of the triangle determined by a first degree function’s (map’s) graphic with the coordinate axis;
7) calculating the distance from the origin of the coordinates system to the graphic of a first degree function (map);
8) determining the intersection point of the graphics of two first degree functions (maps).

Since the mathematics curriculum for ninth grade – to supplement the concepts related to functions (maps) of the second degree function (map) study and the students, having assimilated now, more and more diverse knowledge, you can continue with the following examples:

Examples 8: 1) The area of a square A is a function of the length of its side x. This functional dependence is given by the formula: $A=x^2$;
2) The area of a circle A is a function of the length of its ray r. This functional dependence is given by the formula: $A=\pi r^2$;
3) In Physics, in the uniform accelerated movement, the distance s(t) covered in time t is given by the formula: $s(t)=\frac{a}{2}t^2$, where a is the acceleration.

In high school applications of second degree functions (maps) study must be in the attention of the teacher through the whole chapter. In this way we present the following problem:

Example 9: From a spherical piece of metal one cuts a cylinder having the maximum side area.
Help: One expresses the maximum side area of the cylinder as a function of the sphere’s ray (R) and of the cylinder’s ray (x) and one attaches the second degree function (map): $f(x)=-4y^2+4R^2y$, where $y=x^2$, $x\in[0,R]$ whose graph is a parable and for which the maximum is determined.

We came with this problem because it is a transitional one, from secondary school to high-school, from Geometry to Algebra, from theory to practise and last, but not least, for the variety of knowledge applied for solving it, as side area of the cylinder formula, Pytagora’s theorem, study of the second degree function (map), and solving a second degree equation.

Applications of functions to find and study the equations and obtain inequalities which empirical methods have been proved difficult.

Examples 10: 1) Determine the number of real roots of the equation $x^2-2\ln x+m=0$, where m is a real parametre.
Help: One applies Rolle’s sequence (theorem) for the function (map) $f(x)=x^2-2\ln x+m$, $x\in(0,\infty)$.

Otherwise: graphically represent the following functions: f, g : $(0,\infty) \to \mathbf{R}$, where, for any $x\in(0,\infty)$, $f(x)=x^2-2\ln x$ and $g(x)=n$, $n\in\mathbf{R}$.

2) O. Holder's inequality: if $a_1, a_2, ..., a_n\geq 0$, $b_1, b_2, ..., b_n\geq 0$, $p>1$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}.$$

Help: One consider the function $f(x)=x^\alpha-\alpha x$, $x>0$, where $\alpha\in(0,1)$.

Example 11: Cauchy-Buniakowski-Schwartz's inequality is obtained for p=q=2.

If $a_1, a_2, ..., a_n\geq 0$, $b_1, b_2, ..., b_n\geq 0$, then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right).$$

This inequality can be proved by mathematic induction method, too.

Later, other basic notions in Mathemathics will be defined as functions (maps), too, for example area, volume, geometric transformations, matrice, determinant, sequence, composition laws, etc.

During high-school drawing the graphic of real functions (maps) will be completed by knowledge related to monotony, periodicity, odd – even functions, continuity, and derivability.

In fact, Mathematic Analysis is the one that tackles with the study of real functions (maps) with a real variable. This study is required both by need of describing the evolution of physical processes, technological processes, and economic processes, and by the development of Mathematics itself.

Of course the teacher can take other examples, in the presentation of the concept of function, in economics, geography, the statistics generally in everyday life. As stated above, we here have presented a point of view, and in conclusion, we hope that we formed of the reader an image, even summary on this subject: teaching - learning concept of function in secondary school in Romania.

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