ACTA DIDACTICA NAPOCENSIA

Volume 14, Number 2, 2021 - DOI: 10.24193/adn.14.2.3

FORMATION PROCESS OF COMMON DIVISOR CONCEPT: A STUDY OF REALISTIC MATHEMATICS EDUCATION

Rezan YILMAZ, Merve DÜNDAR

Abstract: Common divisor is one of the concepts that started to be learned in secondary school and forms the basis of many concepts. But students generally have difficulty in making sense of it. The purpose of this study is to investigate concept formation processes of common divisor through a case study on seven sixth grade students. To do this, we designed a learning environment based on Realistic Mathematics Education and tried to determine how students construct the concept. The data collected through group and individual study papers and semi-structured clinical interviews were analyzed with content analysis within the APOS theoretical framework. The findings showed that one student was unable to internalize the action and her thoughts depended on the contextual problem. Other students internalized all common divisors and coordinated them with prime factorization. At the end, students were able to encapsulate all the processes into object by finding the product of common prime numbers. Moreover, students were able to identify greatest common divisor among all common divisors and clarified its meaning.

Key words: Common divisor, Greatest common divisor, Concept formation, Realistic Mathematics Education, APOS

1. Introduction

Mathematics and mathematical thinking are constructed on numbers and their properties, patterns and structures, and built on a conceptual understanding of them. Therefore operations, procedures, relationships and practices on numbers cover an important part in mathematics curriculums taught in schools. Subjects of natural numbers and operations on these numbers which are started to be taught in primary school expand with the concepts of prime number, multiple, divisor, common multiple and common divisor in secondary school (MEB, 2018; NCTM, 2000). These concepts form the basis for the formation of other number sets and the operations on these sets and also the relations of many other concepts in the following years. On the other hand, these concepts are ideal places to learn about problem solving, reasoning, generalization, abstraction and proof, and the problems to be chosen appropriately for these concepts can provide an excellent opportunity for the development and evaluation of mathematical arguments (Selden and Selden, 2002).

Considering that the concepts of multiple, divisor, common multiple and common divisor, like many other mathematical concepts, arise from real-life needs of knowledge and relations, it is thought that teaching these concepts should be realized in a more informal and real-life context (Campbell and Zazkis, 2002; Gravemeijer, 1999). Thus, when students learn by relating with their own lives and performing their own mathematization processes, mathematics will become meaningful for them (Gravemeijer, 1994; 1999; Gravemeijer & Terwel, 2000). However, considering that students have difficulty in understanding and relating mathematical concepts, attention to informal meanings and familiar contexts in mathematics education should not be considered separately from the development of conceptual foundations (Campbell and Zazkis, 2002). In this sense, it is thought that Realistic Mathematics Education (RME) which is based on the environments that the student has actively re-invented the mathematical structures by using his/her own paths and the models he/she has developed, will serve this purpose (Fauzan, Slettenhaar and Plomp, 2002; Gravemeijer, 1999; van den Heuvel-Panhuizen, 2000). There are three stages mentioned in RME (Treffers; 1991b). These are firstly the development of operations-relations in some ordinary real-life contexts, then the realizing the same structure in other contexts, and finally, the formulation of the common structure by symbolizing it. These three steps constitute generalization and abstraction (Mitchelmore, 2002). Considering that the

Received April 2020.

Cite as: Yilmaz, R. & Dündar, M. (2021). Formation Process of Common Divisor Concept: A Study of Realistic Mathematics Education. Acta Didactica Napocensia, 14(2), 30-43, https://doi.org/10.24193/adn.14.2.3

concept is a cognitive structure formed as a result of abstraction (Skiff, 1953; Von Glasersfeld, 1991), the importance of the environment in which abstraction will take place can be noticed in the concept formation process.

Abstraction is directly related with genetic decomposition and mental schemas of concepts. Schema is formed by three components: initial experience of an item that categorized by an observer as trigger or stimulus, an activity that is associated with it, a subsequent experience that is related with the activity like the result of the activity (Von Glasersfeld, 1991). Moreover, Dubinsky (1991) concluded that problem situations, schemas, and responses of learners should be examined for understanding mathematical knowledge and acquisition of it. In this sense, a theoretical framework named APOS (Action, Process, Object, Schemas) was developed for understanding the construction of mathematical knowledge and concepts (Arnon, et al., 2014).

Although it is thought that the research on learning and teaching the concepts that form the basis of the number theory (e.g. primes, factors, multiples, divisors, prime factorization, common multiples, common divisors, least common multiple (LCM), greatest common divisor (GCD)) is very important (Campbell and Zazkis, 2002; Zazkis and Campbell, 2006), it is seen that the studies carried out are still not enough. Also, studies examining how these concepts are abstracted are almost non-existent. When the related studies are examined, the methods to be used for LCM and GCD are mentioned (Sai, 1992; Srivastav, 2016), strategies and difficulties for teaching and learning of GCD are examined (e.g. Davies, 1980; Sai, 1992) and three algorithmic solutions related to GCD in mathematical and computer science problems are conducted (e.g. Lichtenstein, 1999). Most of the other studies have been carried out on prospective teachers. For example, the studies of Zazkis are about the polysemy of divisor and quotients concepts (1998) and about the associating factor, divisor and multiple concepts (2000); the researches of Zazkis and Campbell are on the idea of uniqueness of prime decomposition (1996a) and understanding of divisibility and multiplicative structure of natural numbers (1996b); the study of Campbell (2002) is on the understanding of the concept of division; the study of Brown, Thomas and Tolias (2002) is on the understanding of the concepts of multiplication and division in problems involving divisibility; and the research of Teppo (2002) is about how they developed the divisions of the natural numbers in an environment realized with problem solving activities.

Remembering that the concepts of common divisor and GCD were first learned in secondary school, it is thought that the examination of the formation of these concepts at this level is very important and necessary due to the lack of studies conducted at this level. In addition, it is thought that examining the formation of these concepts by providing a transition environment from informal situations to formal situations through problems based on real life contexts will give important ideas about how these concepts can be learned as well as how it can be taught.

In the below-mentioned theoretical framework, firstly the definition and primary elements of RME have been explained, and information related to its implementation has been presented. Subsequently, APOS theory has also been explained.

2. Theoretical Framework

2. 1. Realistic Mathematics Education (RME)

RME is based on the idea that mathematics is a human activity (Freudenthal, 1973). In this approach, learner reaches formal mathematics knowledge after being abstracted by using his/her informal knowledge in real life by means of re-inventing under the guidance of a teacher (De Lange, 1996; Treffers, 1991a). This transformation is believed to be achieved by contextual problems that are experientially real to the students (Gravemeijer, 1999; van den Heuvel-Panhuizen, 2000). Thus, students can experience mathematization processes that are similar to processes of real mathematicians (Gravemeijer and Terwel, 2000; van den Heuvel-Panhuizen, 2000) and they can construct their own knowledge, which provides mathematics with more meaning. Through mathematization processes, real-life experiences categorized as informal could be formalized (Freudenthal, 1983; van den Heuvel-Panhuizen and Drijvers, 2014).

There are three key principles of RME that could be taken into consideration in implementation and planning of learning as well as teaching process including *guided reinvention through progressive mathematization*, *didactical phenomenology*, and *self-developed models*. Guided reinvention through progressive mathematization requires students to configure and organize the contextual problem by finding mathematical aspects of it (Fauzan, 2002). This research, which is conducted with a strong intuitional component, is considered to be the reinvention of mathematical conception (Yilmaz, 2020). Also, it can be considered as a two-stage process: horizontal and vertical mathematization. Horizontal mathematization includes transforming the contextual problem into a mathematical problem by using informal strategies. During vertical mathematization, students produce a new algorithm and move mathematization process to a higher level in the light of informal strategies and they abstract the conception (Gravemeijer and Terwel, 2000; van den Panhuizen and Drijvers, 2014). According to didactic phenomenology, students should begin events or phenomenon that are meaningful for them. Thereby, they can produce their own phenomenon like real mathematicians (Freudenthal, 1983). Moreover, students can develop their own models with the help of the contextual problem. These self-developed models have a crucial role to pose as a bridge between informal and formal knowledge (Fauzan, 2002; Gravemeijer and Terwel, 2000; van den Heuvel-Panhuizen and Drijvers, 2014). As students engage with the problem, models (model for) that are dependent on the context transforms into the models (model of) independent from the problem situation (Zandieh and Rasmussen, 2010).

Teaching and learning principles of RME that could be important for designing of instruction consist of *constructing and concretizing*, *levels and models*, *reflection and special assignment*, *social context and interaction*, and *structuring and interviewing* (Treffers, 1991a). According to constructing and concretizing principle, students should start to learn with a concrete foundation with rich content problems that support mathematical organization and informal solution processes of the students should be followed in context. Also, teaching should focus on factual discoveries. Levels and models principle means that in the learning process, models and schemes that may arise during the solution process of the problem are used to complete the gaps between students' abstraction levels and to facilitate their cognitive transitions. According to reflection and special assignment principle, special assignments should be given to the student to reveal their free products and thoughts. Thus, as the student develops ideas or models about the problem and thinks deeply about the given problem and his/her own solution processes, he/she can create more advanced mental structures. Social context and interaction principle emphasizes that the effect of group work on learning. So in groups, students who are strong and weak can be in an interactive field. Thus, all students can benefit from collaboration. In addition, as groups share their own solutions and strategies and in-group discussion is supported, group members can discover a different and original solution or new strategies. According to structuring and interviewing principle, if the students are provided to learn the concept in a spiral relation with related prior knowledge, meaningful learning can be provided. Also, knowledge and skills can be constructed with the help of a structured start (Fauzan, 2002; Gravemeijer, 1994; Gravemeijer and Terwel, 2000; van den Heuvel-Panhuizen and Drijvers, 2014).

2. 2. APOS Theory

APOS, which is the abbreviation of the initials of *Action*, *Process*, *Object* and *Schema*, is a theoretical framework for examining concept formation processes. It supports planning teaching, implementation and evaluation process, by allowing researchers and teachers to access students' construction processes and levels (Asiala et al., 1997). According to APOS, a concept is analyzed with mental structures that are action, process, objects and schema and mental mechanisms that are interiorization, coordination, reversal, encapsulation, and thematization for investigating mental formation of students (Asiala et al., 1997; Arnon et al., 2014). Figure 1 illustrates the mechanisms and structures, as well as their relationships.

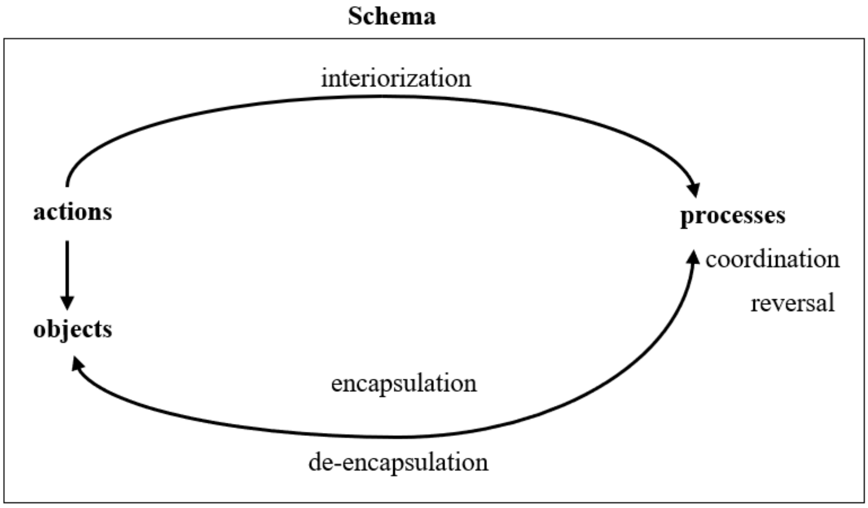

Figure 1. *Mental structures and mechanisms for construction a mathematical concept* (Arnon et al., 2014)

Action is defined as the transformation of previously concieved object(s) (Asiala, et al., 1997). In action, students may calculate or transformate known object(s) with an external stimulus. If students control or reflect the action(s), they can possess process conception by internalizing action (Parraguez and Oktaç, 2010). These internal processes allow students to make sense of perceived phenomena (Dubinsky, 2002). Moreover, students may coordinate two or more processes or reverse a conceived process. Then, they may encapsulate them as an object when it is conceived as a whole. While process is a dynamic structure, object is a static one (Asiala, et al., 1997). Coherent collection of all these actions, processes, and objects construct the schema (Dubinsky, 1991). At the end of the construction process, if the learner is successful, problem or new object has been assimilated by schema. When not successful, his/her existing schema might be accommodated to handle the new phenomenon (Dubinsky, 2002).

Genetic decomposition defines structures and mechanisms that are necessary for the construction of a specified concept (Arnon et al., 2014). It is also defined as a mental model that researchers first make the description of the construction in order to explain the way that students might follow in the construction process (Parraguez and Oktaç, 2010). Thanks to genetic decomposition of the concept, differences and difficulties in acquiring the concept might reveal (Arnon et al., 2014).

There are researches conducted for investigating the construction processes of concepts such as limit, function, vector space, integration, infinity, and congruence with using APOS theoretical framework (e.g. Bergsten, 2008; Dubinsky, Weller, McDonald, and Brown, 2005; Parraguez and Oktaç, 2010; Stalvey and Vidakovic, 2015; Trigueros and Martinez-Planell, 2010; Yilmaz, 2011; Yilmaz and Argun, 2018). There are a few studies examining the formation of number theory concepts with the APOS theoretical framework. For example, Zazkis and Campbell (1996a) gave an APOS analysis of prospective teachers' constructions on divisibility and multiplicative structure of natural numbers. Also, Brown, Thomas and Tolias (2002) investigated how prospective teachers apply their conceptions of multiplication and division to problem situations involving divisibility under APOS theoretical framework.

In this study, 6th grade learners' formation process of common divisor concept which is one of the 'complex cognitive structures' in elementary number theory (Zazkis and Campbell,1996) was examined with APOS theory by trying to define its genetic decomposition. Moreover, as Dubinsky (1991) referred, a problem situation was conducted and examined to understand acquisition of the concept. To achieve this, the learning environment is grounded on the theory of RME. Therefore, the research question of the study is "*how is the formation process of common divisor and GCD concepts of 6th grade students in RME based teaching environment*?".

3. Method

3. 1. Research Design and Participants

A case study was carried out with qualitative research methods in order to examine 6^{th} grade students' common divisor concept formation processes. The study was carried out in a school in a province of the Black Sea region, where one of the researchers was a teacher a while ago. Thus, it was thought that getting to know the school and the students closely would give an opinion about the readiness of the students for the concept and also about their individual and group participation. First of all, two of the three classes in the school were selected by criterion sampling from the purposeful sampling methods in order to carry out the teaching based on RME. The fact that the class is heterogeneous, the students in the class interact and can express themselves well were among the important criteria. One of the selected class had 28 (18 female and 10 male), the other had 29 (16 female, 13 male) students. The same teaching process was carried out in both classes. Before the teaching process started, the students were observed in their mathematics lessons in the classroom for a while, thus, the information about the class and student profiles was enriched and the process was progressed in the natural environment. Later, students were tried to be grouped heterogeneously according to their prerequisite knowledge and their communication skills by having the opinion of their mathematics teacher of both classroom and also by observation of the researcher. In this way, it was tried to ensure that the readiness of the students in the same group about the concept of the common divisor was different and their interactions within the group were at the maximum level. Each group had five or six students in their classes. Participants in the clinical interviews held after the teaching process were selected with the purposeful sampling methods as a maximum variation sampling. Thus, it was tried to select the students who were good at self-expression but with different levels of readiness. During the selection process**,** the opinions of their teacher, observations of the researcher in the teaching process and group worksheets were taken into consideration. After all, seven 6^{th} grade students were determined as participants, five of them were female and two of them were male.

3. 2. Teaching Process and Data Collection

Teaching and learning process was planned according to principles of RME to investigate common divisor concept formation processes. Based on the social context and interaction principle, students were divided into heterogeneous groups according to their level of readiness. It was tried to prepare a natural environment where students could think freely in groups and experience processes similar to those of real mathematicians. In the learning environment where the researcher was a guider, the mathematization processes of the students were tried to be supported. Contextual problems provide students with a structured beginning that helps the formation of the concept, and students can relate the given problem with their existing knowledge and skills. In addition, informal solution strategies, didactic phenomena and the models they create can be revealed in the process of solving problems and creating the concept. In the process of the formation of the concept to be learned, in order to facilitate the mathematical processes, the students were presented with a contextual problem situation that they could have or imagine in real life. This problem has been prepared in accordance with RME by targeting the gains offered by the Ministry of National Education (MEB) in order to examine the processes of forming the concept of common divisor. After the problem was solved in the classroom and in the groups, clinical interviews were conducted.

Before the problem was given, firstly students were asked what they could present to their teacher for the next teachers' day. After their suggestions, students decided to prepare their presents on their own, which could be valuable for their teacher. When students thought about what they could give as a gift to their teacher, the researcher offered them to write their emotions on square sticky notes and then to paste these notes on a clipboard. They discussed how they could use these sticky notes. Also, the researcher asked students if they wanted to use more sticky papers, how they could paste them. Then, it was decided that there would be no space between the notes. And also, the size of the clipboard was decided by considering the suggestions made in the class. After this process, the researcher asked this context as a problem to the students. Opinions from experts in mathematics education and implementation process with a sixth grade student as a pilot study were effective in construction of this problem. You can see the problem used in the class below and a sample visual in Figure 2.

Think that you (all class) will give a gift to your teacher for the next teachers' day. You want to prepare a special gift for your teacher. You decide that the most valuable gift is the one you will prepare with your friends. You write your emotions about your teacher on square sticky notes, then will gift a clipboard that is formed by these notes. Suppose you have an empty rectangular clipboard to fill with your sticky notes in the classroom. If you want no empty space in the clipboard, what side lenght could your identic sticky notes have?

Figure 2. *Sample visual of square sticky notes on a clipboard*

When students discussed the problem in their groups, in-group discussions and different thoughts were supported and students were told that they could write individual and group thoughts on their group worksheet. After students discussed the problem in groups, whole class critisized their thoughts and some of them were written on the board. All the discussions were recorded with a video camera and sound recorder in order to follow the mathematization processes and in-group discussions in detail and also group worksheets were collected. After this process some constructed exercises were also practiced.

Clinical interviews were conducted with selected participants in order to analyze students' mathematization and concept formation processes in depth. In clinical interviews, it was aimed to reveal the cognitive processes of the participants. Therefore, the interviews were held in an environment where the participants feel comfortable and the students were asked to think aloud. In this process, the contextual problem solved in the class was reminded, and what they did while solving this problem, what they thought, what they found, how they found it, and why they thought it was tried to be revealed with semi-structured questions. Thus, the construction of the participant about the concept was tried to be understood by asking to explain the discussions within the group related to the contextual problem. During the clinical interviews, the researcher also asked questions to investigate conceptualization of GCD. These questions included couples of natural numbers that were/weren't multiple of the other, relatively prime numbers and etc. The participants were asked to write down all the process they thought during the interview and the solutions they reached on their individual worksheets. In addition, all clinical interviews were recorded with a video camera and a voice recorder.

3. 3. Data Analysis

In the study, genetic decomposition was prepared based on the APOS theoretical framework in order to analyze how students could construct the concept of the common divisor in their minds. In the process of posing this decomposition, in addition to the mathematical knowledge and experience about the concept of the common divisor, the data of the pilot study conducted with a 6^{th} grade student were also effective. Although it was known that students could realize their constructions in different ways, it was thought that the genetic decomposition posed before the teaching process would help the researchers in preparing the teaching process and materials, as well as providing insight into how students could construct the concept (Parraguez & Oktaç, 2010). Genetic decomposition is given below in Figure 3.

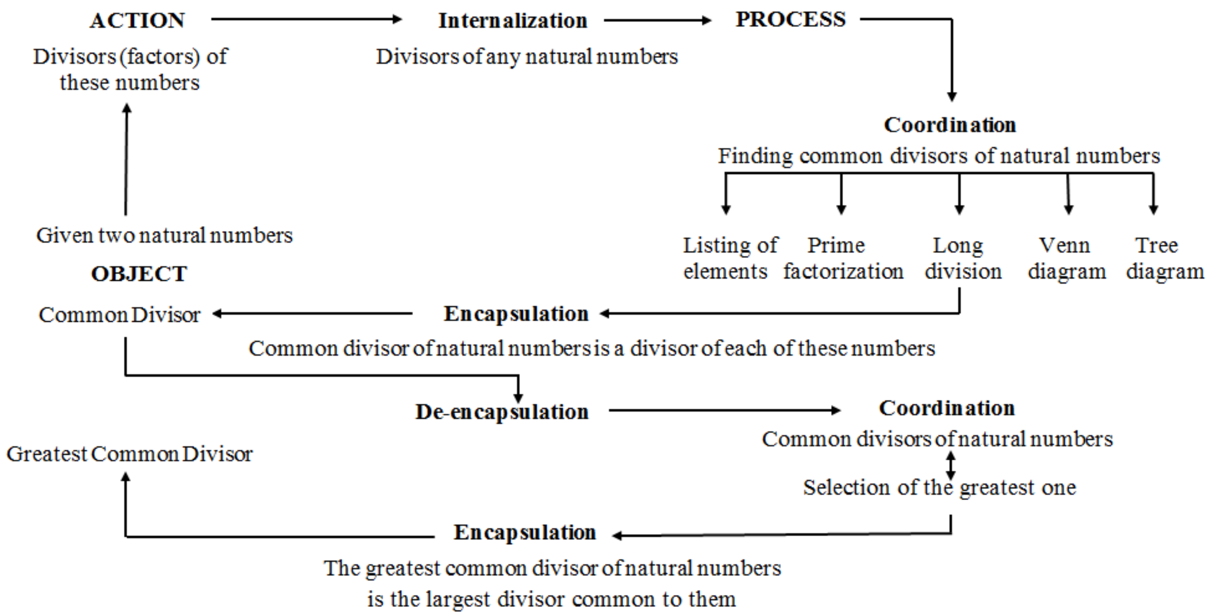

Figure 3. *Genetic decomposition of common divisor and greatest common divisor*

According to genetic decomposition in Figure 3, students start their formations with two given natural numbers. In order to find one side's length of square paper that fully fill the clipboard, they try to find the divisors (factors) of these numbers. That is, in *action*, students could find all factors of the numbers by placing the square papers on the clipboard and by dividing given numbers with the help of a trial and error method. After *internalizing* all divisors (factors) in the process, they could find them systematically. Also, in the *process*, students may *coordinate* the way(s) of finding all common divisors (factors) with methods like prime factorization. Thus, they could systematically find common factors of natural numbers with using this (these) method(s) and use it (them) by making sense. Finally, students can meaningfully *encapsulate* all the process to construct common divisors of any numbers as a (first final) *object*. After students are asked about the sides of the biggest paper, they can *de-encapsulate* all the processes they have executed. They *coordinate* the selection of the biggest with all common divisors, (re)*encapsulate* all the process and then they could reach the GCD as a (second final) *object*.

Transcribed data obtained from the videos and sound recordings was analyzed via the content analysis. After the transcribed data was coded according to APOS theoretical framework, categories were identified by linking different codes (Creswell, 1998; Patton, 1990). For providing reliability of the study, a full consensus has been reached about the formation of data, codes, and categories by researchers. Besides, in order to ensure that the results can be conveyed into similar media, the obtained findings have been supported with the quotations and detailed descriptions have been made (Berg, 2001; Yıldırım and Şimşek, 2006).

4. Findings

The findings related to common divisor and GCD concept were organized within the APOS theoretical framework. The nicknames of the participants are Hakan, Melek, Nurhayat, Gamze, Filiz, Buse, Hilal. All the concept formation processes of the participants were tried to be clarified in this section. Mathematization processes, didactical phenomenon, mental structures and mechanism of the participants were also investigated. Differences in their construction processes constitute main criteria of this research. The findings of the study are tried to be given under two sub-titles related with common divisor and GCD by synthesizing constructions of the participants in their group studies and in clinical interviews.

4. 1. Common Divisor Concept Formation Process

First of all, the contextual problem was given to the students after they were prepared for the problem as mentioned in methodology. Then, a sample cardboard was given to them with sides 18 cm and 30 cm as a clipboard which was prepared in the class together with them. Then, they started to think how they could find the side(s) of square papers in groups. When they started to fold or cut papers, the researcher said that she had square papers with different sides (e.g. 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, and 30) and added that they could use these papers or could form other square papers by cutting the given paper if wanted.

The (starting) *object* was two natural numbers. Actually, these were the sides of the clipboard that was a rectangular cardboard. In group discussions, when students thought about what could be the side(s) of square papers fitting into this cardboard, they tried to place these square papers. Yet, some groups placed different square papers at the same time. Then, they discussed with the researcher about the square papers which had to be identical (congruent).

In clinical interviews, Gamze tried to draw a visual by placing square papers and find which square papers fit into the clipboard. This was the *action* stage of Gamze and in the class, this stage occurred similarly with other participants' constructions. Actually, placing the papers gave them the sense of divisors of the numbers and allowed them to understand the problem that could be actualized in their real life. Then, students thought that they should find the divisors of the numbers. For example, Hakan said, "if the number divides 18, this paper fits into there". Then, students tried to find all divisors of 30 and 18. In some groups, students first found the divisors and then they wanted to place it on the cardboard. In clinical interviews some students, for instance Melek, found all divisors of 30 and 18 systematically as shown in Figure 4. Thus, they may *internalize* how they can find divisors of any natural number.

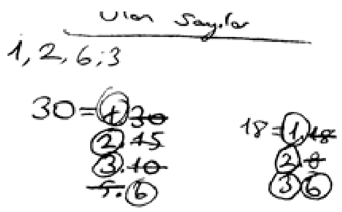

Figure 4. *Melek's study of finding all factors*

Through *internalization*, students showed they were in *process* stage. In group study, one student asked whether a square paper, whose sides were seven, fitted into the cardboard. Then Melek said "7 does not fit because it is not divide neither 18 nor 30". When the researcher asked whether it should divide both of them and why, Melek told "yes, if it were not, there would be a space" and showed it by placing the square paper whose sides 7. Also, Hakan said in the interview "for example 5 is not placed because it divides 30, yet not divides 18. There would be a space on the side of 18". Hilal who was in *action* stage could not pass through *internalization*. For instance, when the researcher asked why they had written "5 cm is proper, she could not clarify the reason, she just said; "my friend found it". Also, she could not explain the reason why they wrote "4 is not proper" (see in Figure 5).

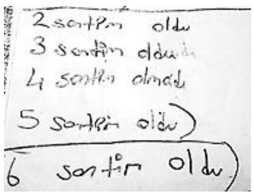

Figure 5. *Action stage of Hilal's group*

When the researcher asked how they found 2, 3 and 6, Hilal said that they found them by dividing, yet she clarified prime factorization method as if they *coordinate* prime factorization by fitting the papers into the clipboard. She explained "firstly we divided 30 by 2 and found the result since 2 times 30 is placed in this paper. Then, the result is 15 when we placed 3 times and then the result is 5 when we placed 5 times" (see the prime factorization in Figure 6). When Hilal said she found 2 and 3 from the prime factorization, the researcher asked how she got 6 and she answered "by multiplying 2 and 3". When the researcher asked the reason, she said "6 is divided by 2 and 3". It could be understood that she just memorized the process that they actualized in the class, yet she could not understand the process.

Figure 6. *Coordinating with prime factorization for Hilal*

Hilal is the only student who was in *action* stage of all the construction. Other students conceptualized the idea. They firstly found common divisors as 2, 3, and 6 by finding all divisors. They prefered the method of listing elements. Then, when the researcher asked them to clarify how to find divisors, they could explain their ways without using the given numbers. For instance, Filiz said that in group study firstly, they checked with her friends whether the numbers were divided by 2. Then she added that they would control whether the other numbers divide both of the given numbers or not. Moreover, when the researcher asked how they could find if the clipboard was much bigger, they tried to find another method in the class. Since they knew to find divisors of the numbers with prime factorization, some students tried this method. They found 2 and 3 were the divisors. When the researcher asked them why they could not find 6, they *coordinated* prime factorization with the method of listing elements and thought how they can reach 6 from 2 and 3. They said the multiplication of 2 and 3 gives 6, yet some of them could not clarify the reason. In the interview, Melek and Hakan could easily *coordinate* these processes by saying how they found 6 without asking and they were faster in operational work. Also, Melek clarified the reason of this by saying "6 is the multiple of both 3 and 2. If a number is divided by both 2 and 3, it is divided by 6 because 6 is divided by both 3 and 2".

Buse, *reversed* the process by checking if 6 divided both of numbers. She said "multiples of 6.., since 3 times 6 is 18 and 5 times 6 is 30, 6 divides both of these numbers". Students *encapsulated* their constructed processes into *object*. For example, Filiz said "for finding all papers that fit into, I signed the numbers that are corresponded within prime numbers, then I found all results by multiplying these". However, because of Hilal's unmeaningful *coordination*, her *encapsuation* of the *coordinated processes* did not make sense. So, she could not construct common divisor meaningfully. Finally, in common divisor concept formation process, students' -didactical phenomenon- was fitting square sticky notes into properly.

4. 2. GCD Concept Formation Process

After the construction of common divisor concept, students were asked to find the biggest papers that fitted into the clipboard. They *de-encapsulated* common divisor concept as an *object*, then easily found 6 as the biggest common divisor. Gamze said "the biggest paper that divides 30 and 18 is 6" and Nurhayat said "6 is the biggest among common divisors". However, Hilal said "the biggest are 5 and 3" (since they are the biggest numbers in prime factors see in Figure 6). Since Hilal was in *action* stage in construction of common divisor concept, she also could not select the biggest one. Also, when the researcher reminded the question's goal, Hilal added "the number 6" by looking at their group paper and by re-thinking the process that they actualized in class.

One of the question that the researchers also asked was about the relationships of the numbers such as 6, 24; 9, 72; 6, 42; 7, 14; and 6, 12 to investigate conceptualization of GCD. Here, among these numbers, one is the multiple of the other. Hakan and Melek investigated these couple numbers and noticed the relationship between them and easily identified that "GCD is the smallest one in numbers that are multiples of each others". Moreover, Filiz, and Nurhayat saw the relationships at the end. Yet, they had difficulties with bigger numbers such as 6, 24; 9, 72 and they had difficulty in operational work as well as stating prime numbers. Also, Gamze said that GCD is the smallest number. Yet, she clarified it as "6 and 7 did not divide them as others, so these are GCDs" and she could not see that the numbers are multiples of each other. Contrary to Gamze, Hilal said "given numbers are multiples of each other", yet she added "6 and 7 are so close to each other" and she failed to understand the relationship.

It was asked the relationship among 7, 9; 7, 8; 15, 8; 25, 9; 15, 4 which are co-prime numbers. Hakan, Melek, and Filiz found the relationship. Filiz said at the end "when the numbers are -so different- from each other, their divisor is different, their common divisor is 1". Since students did not know co-prime numbers, they could not name them but they could just explain them as they understood. Besides, Buse, Nurhayat, and Gamze could not find the relationship. They said "Some of the numbers are prime… They are the numbers that are not multiple with each other… Numbers that are not related". Finally, Hilal said "there is no relationship because common divisors of these numbers are not the same".

5. Discussion and Conclusion

The results of the study demonstrate that the students constructed new structures by relating and organizing their prior knowledge about prime numbers, division, and prime factorization algorithm during formation processes common divisor. By selecting GCD among the common divisors, learners have represented their existing schema in a higher stage, as Dubinsky (1991) explains. However, there are some differences in their construction processes.

Firstly, students found lengths of the sides of square sticky papers that actually fit into the clipboard by placing them. Then, they understood that a side length of the perfectly fitted square sticky paper divided the sides of the clipboard. This gave them the idea of finding all divisors of the numbers. After finding all the divisors, they tried to determine the ones dividing both sides. Actually, for some students, placing papers served to get the idea of divisors. Hovewer in some groups, students firstly found the divisors then they controlled whether it fitted or not by placing square sticky papers. This may stem from the deficiency of reliance of the division method.

The findings identified that students were in process stage by internalizing 'divisors of the numbers'. Some students had difficulty in making operation during the process, so this situation affected their contruction processes negatively. The reason behind that might be the deficiencies in their prior knowledge and the ability of mental calculation. However, students who had good operational ability did not spend much time on finding all divisors and they found the result systematically, rather than by trying all numbers. This enabled them to see the whole process and eased their coordination with the other methods. On the contrary, after understanding the problem, some students overexerted during their operations instead of calculating mentally. Having difficulty of students in mathematical operations during internalization in process stage gives the idea about reflection of their actions, their general explainations or summarizations on the meanings of concepts. So, this makes think that their understanding do not seem strong enough to allow them to respond to inferences rather than actions (Brown, Thomas and Tolias, 2002). Also, sometimes they had difficulty in applying prime factorization and this had negative effect on seeing the whole picture. This shows the necessity and importance of coordinating with prime factorization in the conceptualization of the common divisor. Therefore, prime factorization should be encapsulated meaningfully as a cognitive object. Moreover, some of the students confused multiple and divisor terms in their clarification. However, it is thought that this situation did not result from their deficiency of knowledge, but may be caused by misrepresentation of the terms.

Students sometimes have difficulty in categorizing and synthesizing their learning. So, they can not abstract some properties (Von Glassersfeld, 1991). For example, in finding the relationships of numbers which are the multiples of each others or co-prime numbers, some students could not synthesize or their abstraction process were late. It could be because they experienced difficulty and overexertion in prime factorization and in finding the product of common prime numbers. Moreover, in finding all common divisors, some students had difficulty because of the big numbers of prime factors. Yet, students who did the multiplication of common prime numbers systematically, found all common divisors.

There was only one student who could not conceptualize common divisor. She was in action stage and just memorized the process that they do in groups and applied it. Therefore, she could not apply the same process to find the common divisor for all natural numbers. That is, she could not actualize reflective abstraction process (Sierpinska, 1994). She was dependent on the problem since she always focused on the study papers and her definitions had deficiencies and sometimes were meaningless. Also, her coordinations were either false or deficient. The reason of this situation could be memorization of some operations like prime factorization.

In the process of finding GCD among the common divisors, most students did not have difficulty. This may be the result of their recent knowledge about all common divisors and they have just chosen the biggest one, which referred emprical abstraction. Hovewer, in higher levels, for instance solving problems in GCD (and LCM), students may need to abstract reflectively (Piaget, 2001; Sierpinska, 1994).

While students tried to find the relationship between numbers that are multiple of each other, some of them easily found GCD as the smallest one. These students had conceptualized common divisor meaningfully. Hovewer, some students who had difficulty in operational work and prime factorization, also had difficulty in finding relationship, while some of them found meaningless relationships. Also, since students did not know co-prime numbers, some of the students named them as unrelated numbers and numbers that were not multiple of each other. Moreover some students named these numbers as ones whose common divisor was only 1.

6. Suggestions

In this study, we have investigated 6th grade students' common divisor and GCD concept formation processes in RME based teaching environment. The development of these basic concepts can be investigated for high school and university students as well as for teachers. Thus, it is thought that it can be possible to have a better understanding in a holistic sense about the formation of the concept. In the instruction process of this research, RME was used to reveal the concept formation process. Other approach(s) to investigate the concept formation process may also be used to compare such differences. Thus, suggestions can be made regarding the teaching environment where the formation of the concept can take place meaningfully. Also, construction processes for related concepts such as common multiple and least common multiple can be investigated and suggestions can be made on the relationships of the formation of these concepts.

References

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller K. (2014). *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*. New York: Springer.

Asiala, M., Brown, A, DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1997). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. Schoenfeld ve Dubinsky (Eds.), *Research in Colllegiate Mathematics Education II, CBMS Issues in Mathematics Education* (Vol 2, s. 1-32). American Mathematical Society, Providence, RI.

Berg, B. L. (2001). *Qualitative research methods for the social sciences* (4th Ed.). Boston: Allyn & Bacon.

Bergsten, C. (2008). On the influence of theory on research in mathematics education: the case of teaching and learning limits of functions. *ZDM*, *40*(2), 189-199.

Brown, A., Thomas, K., & Tolias, G. (2002). Conceptions of divisibility: Success and understanding. In S.R. Campbell and R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*, (pp.41-82). London: Ablex Publishing.

Campbell, S. R. (2002). Coming to Terms with Division: Preservice Teachers' Understanding. In S. R. Campbell and R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*, (pp.15-40). London: Ablex Publishing.

Campbell, S. R. & Zazkis, R. (2002). Toward Number Theory as a Conceptual Field. In S. R. Campbell and R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*, (pp.1-14). London: Ablex Publishing.

Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions.* Thousands Oaks, CA: Sage.

Davies, H. B. (1980). On highest common factor and least common multiple in the secondary school mathematics curriculum, *International Journal of Mathematical Education in Science and Technology*, *11*(1), 115-122.

De Lange, J. (1996). Using and applying mathematics in education. In A. J. Bishop et al. (Eds.). *International handbook of mathematics education (*49-97). The Netherlands: Kluwer Academic Publishers.

Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In *Epistemological foundations of mathematical experience* (pp. 160-202). Springer, New York, NY.

Dubinsky, E. (2002). Reflective abstraction in advanced mathematical thinking. In *Advanced mathematical thinking* (pp. 95-126). Springer, Dordrecht.

Dubinsky, E., Weller, K., Mcdonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An APOS-based analysis: Part 1. *Educational studies in mathematics*, *58*(3), 335-359.

Fauzan, A. (2002). *Applying realistic mathematics education (rme) in teaching geometry in Indonesian primary schools*. Unpublished doctoral dissertation, Universiteit Twente, Indenesia.

Fauzan, A., Slettenhaar, D., & Plomp, T. (2002). Traditional mathematics education vs. realistic mathematics education: Hoping for changes. In P. Valero and O. Skovsmose (Eds.). *Proceedings of the 3rd International Mathematics Education and Society Conference* (pp. 1-4). Copenhagen: Centre for Research in Learning Mathematics.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht: Reidel Publishing.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Riedel Publishing.

Gravemeijer, K. (1994). *Developing realistic mathematics education.* Utrecht: CD-β Press.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155-177.

Gravemeijer, K. & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777-796.

Ministry of Education [MEB]. (2018). *Mathematics Curriculum (1-8),* Ankara: Ministry of Education Publishing.

Lichtenstein, O. (1999). Thoughts about the greatest common divisor problem in the context of mathematical problem solving and computer science. *Mathematics and Computer Education*, *33*(2), 171.

Mitchelmore, M. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. Paper presented at the *Proceeding of The East Asia Regional Conference on*

Mathematics Education (2nd) and *The Southeast Asian Conference on Mathematics Education* (9th, Singapore, May, 27–31).

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Parraguez, M. & Oktaç, A. (2010). Construction of the vector space concept from the viewpoint of APOS theory. *Linear Algebra and its Applications*, *432*(8), 2112-2124.

Patton, M. Q. (1990). *Qualitative evaluation and research methods*. Sage Publications.

Piaget, J. (2001). *Studies in reflecting abstraction*. R. L. Campbell (Ed.). USA: Psychology Press

Sai, P. K. (1992). Reflection on HFC and LCM: variety of mathematical connections. *Teaching and Learning*, *13*(1), 41-50.

Selden, A. & Selden, J. (2002). Reflections on Mathematics Education Research Questions in Elementary Number Theory. In S. R. Campbell and R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*, (pp.213-230). London: Ablex Publishing.

Sierpinska, A. (1994). Understandings in Mathematics. London: Falmer.

Skiff, S. (1953). Concept formation and education. *Peabody Journal of Education*, *30*(5), 296-299.

Srivastav, M. K. (2016). H.C.F and L.C.M – A Case Study of Completeness Property of Real Numbers. *International Journal of Mathematics Trend and Technology (IJMTT)*, *30*(1), 30-33.

Stalvey, H. E. & Vidakovic, D. (2015). Students' reasoning about relationships between variables in a real-world problem. *The Journal of Mathematical Behavior*, *40*, 192-210.

Teppo, A. R. (2002). Integrating Content and Process in Classroom Mathematics. In S. R. Campbell and R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*, (pp.117-130). London: Ablex Publishing.

Treffers, A. (1991a). Didactical background of a mathematics program for Primary Education. In L. Streefland (Eds.), *Realistic Mathematics Education in primary schools* (pp. 21-57). Utrecht: Freudenthal Institute, Utrecht University.

Treffers, A. (1991b). Realistic mathematics education in the Netherlands 1980- 1990. In L. Streefland (Ed.), *Realistic Mathematics Education in primary schools.* Utrecht: Freudenthal Institute, Utrecht University.

Trigueros, M. & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational Studies in Mathematics*, *73*(1), 3-19.

Van den Heuvel-Panhuizen, M. & Drijivers, P. (2014). Realistics Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 521-525). Dordrecht: Springer.

Van den Heuvel-Panhuizen, M. (2000). Mathematics education in the Netherlans: a Guided Tour. Freudenthal Institute Cd-rom for ICME9. Utrecht: Utrecht University.

Von Glasersfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget's approach. In *Epistemological foundations of mathematical experience* (pp. 45-67). New York, NY: Springer.

Yıldırım, A. & Şimşek, H. (2006). *Qualitative Research Methods in Social Sciences,* Ankara: Seçkin Publishing.

Yilmaz, R. (2011). Visualization in mathematical abstraction and generalization processes and its role. Unpublished doctoral thesis. Ankara: Gazi University.

Yilmaz, R. & Argun, Z. (2018). Role of visualization in mathematical abstraction: The case of congruence concept. *International Journal of Education in Mathematics, Science and Technology*, *6*(1), 41-57.

Yilmaz, R. (2020). Prospective mathematics teachers' cognitive competencies on realistic mathematics education. *Journal on Mathematics Education*, *11*(1), 17-44.

Zandieh, M. & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*(2), 57-75.

Zazkis, R. (1998). Divisors and quotients: Acknowledging polysemy. *For the Learning of Mathematics, 18*(3), 27–30.

Zazkis, R. & Campbell, S. (1996a). Prime decomposition: Understanding uniqueness. *The Journal of Mathematical Behavior, 15*(2), 207-218.

Zazkis, R. & Campbell, S. (1996b). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education*, *27*(5), 540-563.

Zazkis, R. (2000). Factors, divisors and multiples: Exploring the web of students' connections. *Research in collegiate mathematics education, 4,* 210-238.

Zazkis, R. & Campbell, S. (2006). *Number Theory in Mathematics Education: Perspectives and Prospects*. Mahwah, NJ: Lawrence Erlbaum Associates.

Authors

Rezan YILMAZ, Assos. Prof. Dr., Ondokuz Mayis University, Faculty of Education, Department of Mathematics and Science Education, Samsun (Turkey). E-mail: rezzany@omu.edu.tr

Merve DÜNDAR, Mathematics Teacher, Ministry of National Education, Yomra Hafız Murat Köseoğlu Secondary School, Trabzon (Turkey). E-mail: merve.dndr1@gmail.com