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LEARNERS' PHILOSOPHY OF MATHEMATICS IN RELATION TO THEIR MATHEMATICAL ERRORS

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Abstract: This paper discusses the philosophies of mathematics in relation to learners' errors and misconceptions in mathematics. The paper argues that it is crucial for teachers to determine the mathematics philosophical bias of their learners. Such an understanding helps teachers to find ways to convince learners to re-consider their philosophies of mathematics if their current philosophy hampers their mathematics learning. The researcher argues that although different, the different philosophies of mathematics are all applicable at various stages of mathematics and of mathematics teaching and learning. In the main, it is important to acknowledge that many mathematical concepts already exist outside many learners knowledge and it is the duty of learners to acquire them. However in learning these existing mathematics concepts, learners have to re-construct this mathematical knowledge through a fallible human process prone to formation of pre-conceptions and alternative conceptions which are not correct. The paper argues that the philosophy of mathematics that a learner ascribes influences how they learn mathematics and therefore the misconceptions and errors they are likely to form.

Keywords: philosophy of mathematics, learner errors

1. Introduction

The paper puts it that different learners hold different philosophies of mathematics even though they may not be expressly clear to them. The author argues that an understanding of the philosophical biases of mathematics learners helps in the understanding of the errors and misconceptions the learners are prone to. The paper aims to discuss the different philosophies of mathematics and how they relate to the mathematics errors and misconceptions that learners are likely to have. The paper was written in the context of a study on learner errors to differential calculus examination items.

2. Background

Mathematical scholars hold quite divergent and competing views on the nature of mathematics; the philosophy of mathematics (see Lakatos, 1976; Polya, 1973; Hersh, 1997; Ernest, 1985, 1991, 1995). According to Korner (1986), the philosophy of mathematics is a reflection upon mathematics and does not add or increase the amount of mathematical knowledge per se. Rather it gives an account of mathematics. In contemporary times, Ernest (1985, 1991, 1995) has argued that the philosophy of mathematics concerns the epistemological and ontological basis of mathematics which are an important educational discipline. Epistemology is the study of what knowledge is and how we come to know, whereas ontology concerns the study of the nature of reality, of being (Ernest, 1991). Thus epistemology also concerns the methodology of obtaining the knowledge that we believe is real knowledge. The philosophy of mathematics influences the type of mathematical knowledge that goes into the curriculum and how it may be taught and learnt; its methodology.

The nature of mathematics goes beyond the conceptual and procedural knowledge of mathematics (Eves, 1990 quoted by Osei, 2000). The knowledge about the nature of mathematics influences how one learns mathematics and what to that person, is important in mathematics and is not. For example, students fixated with the belief that the essence of mathematics is the ability to efficiently carry out calculations (the arithmetic view), are bewildered with the onset of algebra, which they at first do not

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realise makes it possible for them to generalise calculations in a much more powerful and elegant way. In the same way students who believe that mathematics is about application of formulae, become very keen at memorising formula. Such learners are not bothered by the inductive investigations that generate the formula; they are interested only in the deductive application of the formulae in order to obtain answers to questions. As students have varied beliefs about what matters in mathematics (in particular students replicate the beliefs of their former teachers), they tend to have different errors in mathematics related to their beliefs. So the belief system adopted by a leaner may influence the errors and misconceptions they may have.

The researcher argues that in doing a study of errors in mathematics, and calculus in particular, a rigorous scholarly discussion of the nature of mathematics needs to be discussed. The philosophy of mathematics incorporates perspectives and belief systems about mathematics.

According to Ernest (1985) philosophy is of value for the teaching of mathematics in that it influences what mathematics is taught, its nature, and how it is taught and learnt. He argues that there are two main movements in the philosophy of mathematics: absolutists on one extreme; and Fallibilists on the other. The absolutist philosophies of mathematics include logicism, formalism and Platonism. These generally agree that mathematical knowledge is absolute, immutable and has its own existence independent of the knower. This group argues that mathematical knowledge is certain, objective and true. The duty of humans is to discover its truth. However the constructivists hold that mathematics is a human creation, made by man to solve the practical and theoretical problems met in man’s existence (Lakatos, 1976; Polya, 1973; Hersh, 1997). Indeed they argue, mathematics is not the absolute truth. Being created by man, it is prone to error as humans are prone to error. Constructivist epistemology explains the nature of knowledge and how human beings learn. Constructivism encompasses that mathematics is a fallible. Closely allied to Fallibilism is Intuitionism. In this research I assume that learners learn mathematics through constructivism, in that learning they are fallible, and so hold misconceptions of absolutist principles of mathematics. I argue that learning occurs through cognitive constructions but that such constructions are also misconstructions, which are misconceptions. Thus both methodology of constructivism underlie this study even though I disagree that mathematical concepts themselves are fallible. What is fallible and fully human is having errors and misconceptions in the journey of acquiring mathematics concepts. I briefly outline the philosophies of mathematics.

3. Philosophies of mathematics

Five philosophies of mathematics will be discussed. These are Platonism, Formalism, Logicism, Intuitionism and Fallibilism.

The first philosophy of mathematics discussed is Platonism (Frege, 1884 see Dummett, 1991). This is the classical philosophy of mathematics. Ernest (1985) observes that Platonists (term named after Plato a classical Greek philosopher) regard mathematical objects as having an ideal existence in some abstract world. For example numbers are regarded to exist in their own world somewhat metaphysically. The Pythagoreans were early Platonists and regarded numbers mythically (Eves, 1990). The Pythagoreans where shattered to discover that the square root of two was not rational, in that it could not be expressed as a ratio of two whole numbers to the point that one of their members who showed this amazing property of number was put to death as this new discovery threatened their strongly held beliefs of number. They were afraid that their member will spread that heresy which would threaten their very being. Platonists discard as nonsensical the suggestion that human beings construct mathematical knowledge, rather, they argue, humans discover mathematical knowledge which exists independent of their consciousness.

Platonists believe in a rigid and fixed body of knowledge as opposed to the dynamic nature of mathematical knowledge argued for by constructivist philosophers. Platonists argue that knowledge can be reduced to form different disciplines such as mathematics, geography, natural sciences and history for example. Platonism has been the main philosophy of mathematics up to the 20th century despite the blows shacking it to its very foundations through first the invention of irrational numbers (first crisis of mathematics) and the invention of complex numbers (Eves, 1990), the invention of non-Euclidean geometries by Gauss, Bolyai, Lobachevsky, Riemann ((see Gray, 2004, Martin, 1991) (second crisis of mathematics) and contradictions discovered when mathematics was reduced to laws of logic by Frege (third and greatest crisis of mathematics) (Dummett, 1991). Although there are still many Platonists, they meet problems on explaining how mathematics should be taught, primarily that their philosophy seems to imply that teaching mathematics ought to be through telling already existing facts. If this view is to be of help to mathematics education, then it is necessary for teachers to diagnose what problems learners are meeting to acquire that fixed knowledge that they are yet to discover. Thus learner mathematics error diagnosis and analysis becomes important in teaching and learning immutable mathematics knowledge.

The second philosophy of mathematics discussed is logicism (Frege see Dummett, 1991, Russell, 1919). Logicism is the view that mathematics is part of logic and can be reduced to logical laws (Russell, 1919). Russell argued that mathematics can be reduced to simple laws of logic. As such, he argued, mathematics can be derived from rules, axioms and postulates, from which all mathematical results can be obtained through careful and rigorous reasoning. Axioms and postulates are mathematical statements that are regarded as patently true and do not need to be proven. An example is Euclid's postulate that the shortest distance between any 2 points is a straight line. However, Russell (1919) discovered that these simple logical laws were sometimes inconsistent and contradictory leading to the third crisis in mathematics, for example Euclid shortest distance between 2 points postulate does not hold in spherical geometry, as the shortest distance between any two cities on the globe is not a straight line, but lies on a great circle. The crisis caused by supposedly consistent laws lead to a revision of logicism as the nature of mathematics. To some extent, this view of mathematics is applicable in this study, particularly on the definition of the derivative although no axioms or postulates are involved. The basis of the derivative, and hence calculus emanates from rigorous reasoning, in the definition of the derivative that involves the very abstract concept of a limit. Calculus then is built on the foundations of logic being applied to functions that are geometrically represented.

The third philosophy of mathematics discussed is Formalism whose principal proponent was David Hilbert (1862-1943) (see Snapper, 1979). "Formalism is the view that mathematics is a meaningless game played with marks on *paper, following rules"* Ernest (1985, p.606). To formalists, mathematics is a strictly formal logical system that is based on postulates and axioms used to prove theorems. There is no concern for meaning. To formalists, axioms and postulates are the basis of mathematics, not contexts. The formalists deny meaning to all mathematics claiming that mathematics is all about symbols and rules for manipulating them. This philosophy of mathematics urges for form without substance and seems anathema to mathematics teaching and learning with understanding as we know it. No lasting mathematical learning can be obtained from such a philosophy where mathematics is studied independent of contexts and meaning. Mathematics problem solving is impossible with this philosophy. Teachers who are poorly educated in mathematics often subscribe to this philosophy of mathematics. Formalism can be compared to procedural teaching and learning, which is concerned with use of "rules without understanding" (Skemp, 1978) or use procedures without connection to meanings or contexts (Stein et al., 2003). I argue that learners who subscribe to this philosophy of mathematics often hold many misconceptions in mathematics because they have a very superficial understanding of what mathematic is. This researcher does not subscribe to this elitist view of mathematics.

The fourth philosophy of mathematics discussed is Intuitionism brought forward by Brouwer (1881–1966). This philosophy is opposes Platonism. Intuitionists base mathematics on people's beliefs and insight occurring in their minds. They regard intuition as critical in the formation of mathematical proofs for example. Their argument is that mathematics exists in the minds of people. The intuitionists heavily suspect classical, absolutist mathematics which they regard as quite unsafe. Intuitionists believe that no mathematical knowledge could exist beyond that which the mathematician has proved. I do not subscribe to this subjective view of the nature of mathematical knowledge. To suggest no mathematics exists beyond what one has proven is to ignore the social nature of mathematical knowledge where mathematical claims are subjected rigorous tests by other mathematicians in a sphere of sharing knowledge. The importance of this philosophy to this study is that often learners make errors and misconceptions emanating from their intuitions. Intuition then is very important as it can explain how learners come to intuitively have alternative conceptions. Indeed intuitions which come from generalising correct mathematical results are major source of mathematical errors and misconceptions. Such intuitions are quickly generalised by without thorough checking for possible loopholes. This results in many mathematical misconceptions.

I presume that adopting intuitionism is good for a start as teachers suspend the conclusive mathematical results while they wait to for the learner to re-examine their intuitions before accepting publicly validated mathematical truths. Intuition is also a very important strategy for coming up with conjectures that can then be rigorously investigated before they are corroborated. Intuition drives inductive reasoning in mathematics but to argue that there are no true mathematical results that exists if we have not personally proved them to me seems to cynical.

The last and fifth philosophy of mathematics discussed is Fallibilism. Fallibilism is one of the latest philosophies on the nature of mathematics to counter absolutism (Lakatos, 1976; Polya, 1973; Hersh, 1997). Fallibilism is a constructivist philosophy arguing that people construct knowledge in their heads including mathematical knowledge. Thus all mathematical knowledge is the product of human creation, the product of their intelligence and hence cannot exist independent of them as Platonists argue. Fallibilism regards the socio-cultural and historical development of mathematics as paramount in explaining the nature of mathematics. Fallibilism argues that knowledge does not need perfect evidence. Fallibilism regards the development of mathematics as spurred on by the recognition that some long established results in mathematics had errors (Eves, 1990). A case in point is Pythagorean discovery that $\sqrt{2}$ is not rational. This was the first crisis in mathematics. This crisis showed distinctly that mathematical formulations are prone to errors and misconceptions even to the most talented and able mathematicians like the Pythagoreans. As already referred to, Euclid's postulate that the shortest distance between any two points is a straight line was rebuffed by Lobachevsky (Eves, 1969). For centuries mathematicians held fast this view as incorrigible and unquestionable until Lobachevsky introduced lunar geometry. Then Euclid's postulate suddenly was exposed. Similarly, Gödel's theorems proved that mathematics is not ontologically incorrigible and Russell (1919) showed that mathematical truths are essentially contradictory in the third crisis of mathematics. Radical fallibilists regard that mathematics is as fallible as science.

Fallibilism values how mathematical knowledge is historically created through informal means such as trial and error. Fallibilists' views of mathematics encompass enthomathematics and are an increasingly powerful in the teaching and learning mathematics, as regards equitable access to mathematics (Boaler, 2003). They take into account the historical and cultural identities of learners, which mean that learners' diversities are respected. This research embraces Fallibilism in learning mathematics and acknowledges learners' making mathematical errors, evidence of their fallibility, as perfectly acceptable and natural.

4. Discussion

As in the preceding discussion, the nature of mathematics; its ontology, is regarded differently by different scholars (Ernest, 1985, 1991, 1995; Lakatos, 1976; Polya, 1973; Hersh, 1997; Russell, 1919) . These perspectives lie in a continuum. Platonists on the extreme right conclude that mathematics is objective, ahistorical, immutable truth existing independent of consciousness. On the extreme left, Fallibilists argue that mathematics is subjective; socio-cultural, error prone, and that no mathematical truth can exist outside what people have consciously constructed. Thus to Fallibilists, mathematics lies in the heads of mathematics practitioners and the mathematics texts that they produce and no were else.

The researcher argues that these different philosophies of mathematics provide important lens to study learner errors and misconceptions in mathematics and their relationship to the different philosophies of mathematics learners might personally hold. In my opinion, each of the philosophies have their rightful place in the teaching and learning of in mathematics. Each has its strengths as well as weaknesses. The researcher believes that all the philosophies are applicable to different aspects of mathematics, as well as its teaching and learning. I argue that the process of learning mathematics is Fallibilist in that when students learn mathematics they do so by constructing mathematics concepts in their own minds. When learners are constructing mathematics concepts, they do use intuition and also trial and error when they conjecture mathematical results and check them out before they establish and affirm them. I argue that when learners are constructing concepts, they sometimes construct concepts that are incomplete, amateur concepts, alternative conceptions, pre-conceptions and transitional concepts. Such constructed concepts may be correct, partially correct or completely wrong. Thus the learners' construction of mathematics concepts is a fallible process. My argument is that if we say something is fallible, that implies that there is something that is correct that may not be fallible. Therefore fallibility is a natural process in learning mathematics. That there are some universal mathematics concepts may not be disputed. For instance, in any language, there are words to relate to one or two for example. One and two are mathematical concepts that all humans have, so that implies that one and two are already there in every person's culture. A very young learner may not have yet been understood the concepts of one and two, yet there already exist. However if we take concepts like triangle, that is likely to be a concept invented by a certain group of people to understand and master their environment. That concept might not be present in other groups of people. So some maths concepts are latent, ever-present such as one or two. Yet some are invented by people for example the imaginary number *i.* Yet more are discovered; such as the discovery of irrational numbers by Pythagoreans.

Logicism is important in mathematics as logical reasoning, both inductive and deductive, are critical in undertaking mathematical discourse. If students have loopholes in their thinking, they perform logical errors. Again intuition is very important in doing mathematics. Often when learners are solving a mathematics problem they may think of some ideas or clues that can be useful in solving that problem. This occurs despite the fact that at first learners cannot clearly reason why they think the clue will be helpful. When such intuition and imagination is lacking in learning mathematics, students often fail to perform well in mathematics. The use of mathematical symbols and notation involves formalising mathematics ideas in how they are written down and communicated in the mathematics community. Mathematical notation is so common that the same mathematical symbols are used in different languages. The failure of students to understand formal mathematical notation results in their failure to understand important mathematical concepts. The lack of understanding of mathematical notation results in syntactical errors and misconceptions. Therefore, the varied philosophies of mathematics provide important avenues to understand why learners have the errors and misconceptions they have. Therefore the philosophies of mathematics that students are inclined to provide important ways to understand learners' errors particularly during interviews when learners' preferred philosophies of mathematics may be gleaned. For teachers, it is also important to consider learners current philosophy. This will enable teachers to help students more as they begin to understand the world-view of mathematics from the learner's point of view. Knowledge of learner philosophies of mathematics is helpful to teachers as a learner's philosophy might have to be modified, if not currently helpful. That way, learners may be encouraged to embrace other philosophies that help them to understand mathematics and concepts better. For example a learner with a Formalist philosophy of mathematics might consider mathematics as a subject composed of meaningless symbols which have to be remembered and manipulated correctly to get the correct answer. Such a leaner needs to be helped to understand that mathematics is a subject full of meaning that is useful for solving daily life problems.

The researcher concludes that philosophies of mathematics are important in doing research on learner errors and misconceptions in mathematics as such philosophies shed light in learners' thinking about what mathematics is. That understanding is crucial in understanding how learners conceive of their errors in mathematics.

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