Volume 6, Number 2, 2016
6. kötet, 2. szám, 2016

TEACHING AND LEARNING HYPERBOLIC FUNCTIONS (I); DEFINITIONS AND FUNDAMENTAL PROPERTIES

Teodor Dumitru Vălcan

Abstract: We propose that in some papers (4 maybe 5!) to present a model of teaching and learning in secondary education of hyperbolic functions, giving many properties (starting with fundamental ones and finishing with development in Tayler series), and then to present some of their applications in algebra and mathematical analysis. The papers addresses teachers in the teaching of these issues in the classroom, but also in preparing students to the school competitions.

Key words: hyperbolic functions, hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, hyperbolic cosecant.

1. Introduction

This paper presents a didactic exposition of fundamental properties of so-called "*hyperbolic functions*" in many aspects analogous to the usual trigonometric functions.

Hyperbolic functions meet together many times in different physical and technical research, having a very important role in non-Euclidean geometry of Lobacevski participated in all relationships (interdependencies) this geometry, see [7]. But independently of these annexes, the theory of hyperbolic functions can present a significant interest to a student or a teacher of Mathematics in secondary education because the analogy between the hyperbolic and trigonometric functions clarifies in a new face many problems of trigonometry.

I shall not dwell here on matters pertaining to the history and philosophy appearance hyperbolic functions, but rather will play their fundamental properties, but also to each to other hyperbolic functions, and we will explore some interesting applications of these functions.

Hyperbolic functions occur naturally as simple combinations of exponential function, e^x, a function that is much studied in School Mathematics. Indeed, the two main functions, hyperbolic cosine, and hyperbolic sine is semisum or semidifference of e^x and e^{-x}, see the following equalities (1.1) and (1.2).

In undergraduate education in Romania, these functions are almost unknown, so students and teachers, despite the fact that they present many similarities to the trigonometric functions and, in addition, have numerous applications in integral calculus. I must admit that no students from the Faculty of Mathematics do not really know these things. In conclusion, we can say that literature the domain in Romania is very poor in providing information about the hyperbolic functions and their applications. Abroad, things seem to be any other way. There are papers, books and websites dedicated exclusively to these functions (see References). For example, the database http://mathworld. wolfram.com to Section HyperbolicFunctions.html, presents eight such books that have chapters or relates entirely to hyperbolic functions; see [1], [2], [3], [4], [5], [6] [8] and [9]. Work in the field, published on the internet are of very poor quality, showing them more trivial formulas mathematics and philosophy, having no a high degree of "*scientific value*". A person eager for knowledge does not really have what learn from these works (see [10] - [16].) In [10], [11], [12], [13] and [15] just find formulas without proof, or with or incomplete proofs. In [16] find some elementary exercises and in [14], Shutz, Am. L., in *Hyperbolic Functions, Expository Paper, Masters Thesis*, presents some basic matters, graphics and immediate applications of hyperbolic functions. We mention the fact that there are many videos on YouTube showing some lessons about hyperbolic functions and their applications.

Naturally, in such circumstances, you shall say: to what uses this work or others, such as this, if we

Received: 15 August 2016

find so much information about these functions on the Internet?

The answer is simple: Because any other relevant information are shown here in a new and special way. In fact, we want to present the reader attentive and interested in these issues, a model of teaching and learning of these functions in high school.

2. The definitions of hyperbolic functions

In this first section we define the hyperbolic functions and present the first their properties.

Definition 2.1: *The function sh :* $\mathbf{R}\to\mathbf{R}$*, given by law, for every* $x\in\mathbf{R}$,

$$sh(x)=\frac{e^{x}-e^{-x}}{2}, \tag{2.1}$$

is called ***hyperbolic sine (in latin, sinus hyperbolus)*** *by argument x.*

Definition 2.2: *The function ch :* $\mathbf{R}\to[1,+\infty)$*, given by law, for every* $x\in\mathbf{R}$,

$$ch(x)=\frac{e^{x}+e^{-x}}{2}, \tag{2.2}$$

is called ***hyperbolic cosine (in latin, cosinus hyperbolus)*** *by argument x.*

Definition 2.3: *The function th :* $\mathbf{R}\to(-1,1)$*, given by law, for every* $x\in\mathbf{R}$,

$$th(x)=\frac{sh(x)}{ch(x)}, \tag{2.3}$$

is called ***hyperbolic tangent*** *by argument x.*

Definition 2.4: *The function cth :* $\mathbf{R}^{*}\to(-\infty,-1)\cup(1,+\infty)$*, given by law, for every* $x\in\mathbf{R}^{*}$,

$$cth(x)=\frac{ch(x)}{sh(x)}, \tag{2.4}$$

is called ***hyperbolic cotangent*** *by argument x.*

Definition 2.5: *The function sch :* $\mathbf{R}\to(0,1]$*, given by law, for every* $x\in\mathbf{R}$,

$$sch(x)=\frac{1}{ch(x)}, \tag{2.5}$$

is called ***hyperbolic secant*** *by argument x.*

Definition 2.6: *The function csh :* $\mathbf{R}^{*}\to\mathbf{R}^{*}$*, given by law, for every* $x\in\mathbf{R}^{*}$,

$$csh(x)=\frac{1}{sh(x)}, \tag{2.6}$$

is called ***hyperbolic cosecant*** *by argument x.*

It is necessary here the following remarks:

Remarks 2.7: *From the above definitions it follows that*:

1) *The functions sh and ch are linear combinations of exponential functions*

$x\mapsto e^{x}$, *respectively* $x\mapsto e^{-x}$, $x\in\mathbf{R}$;

and vice versa, that is, the functions e^{x} *and* e^{-x} *are linear combinations of the functions*

$x\mapsto sh(x)$, *respectively* $x\mapsto ch(x)$, $x\in\mathbf{R}$;

because, for every $x\in\mathbf{R}$:

$$e^{x}=ch(x)+sh(x) \tag{2.7}$$

and

$$e^{-x}=ch(x)-sh(x). \tag{2.8}$$

2) *For every* $x\in\mathbf{R}$,

$$sh(x)=\frac{e^{2x}-1}{2e^{x}}=\frac{1-e^{-2x}}{2e^{-x}}. \tag{2.1′}$$

3) *For every* $x\in\mathbf{R}$,

$$ch(x)=\frac{e^{2x}+1}{2e^{x}}=\frac{1+e^{-2x}}{2e^{-x}}. \tag{2.2′}$$

4) *For every* $x\in\mathbf{R}$,

$$th(x)=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{e^{2x}-1}{e^{2x}+1}=\frac{1-e^{-2x}}{1+e^{-2x}}\,. \tag{2.3′}$$

5) *For every* $x\in\mathbf{R}^{*}$,

$$cth(x)=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=\frac{e^{2x}+1}{e^{2x}-1}=\frac{1+e^{-2x}}{1-e^{-2x}}\,. \tag{2.4′}$$

6) *For every* $x\in\mathbf{R}$,

$$sch(x)=\frac{2}{e^{x}+e^{-x}}=\frac{2e^{x}}{e^{2x}+1}=\frac{2e^{-x}}{1+e^{-2x}}\,. \tag{2.5′}$$

5) *For every* $x\in\mathbf{R}^{*}$,

$$csh(x)=\frac{2}{e^{x}-e^{-x}}=\frac{2e^{x}}{e^{2x}-1}=\frac{2e^{-x}}{1-e^{-2x}}\,. \tag{2.6′}$$

6) *To make analogies with the trigonometric functions, but for abbreviations, we use the following notations further*:

a) *for every* $x\in\mathbf{R}$,

$sh(x)\overset{not.}{=}shx$, $ch(x)\overset{not.}{=}chx$, $th(x)\overset{not.}{=}thx$, $sch(x)\overset{not.}{=}schx$,

b) *for every* $x\in\mathbf{R}^{*}$,

$cth(x)\overset{not.}{=}cthx$, $csh(x)\overset{not.}{=}cshx$;

but any expression including simple fraction, which will be the argument of one of these functions will be put between brackets. Also here we specify that in some papers functions: sh, ch, th, cth, sch, csh, they are denoted, respectively: sinh, cosh, tanh, cotanh, sech, cosech. □

Remarks 2.8: *From the above definitions and remarks it follows that*:

1) *The function sh is odd, i.e.: for every* $x\in\mathbf{R}$,

$$sh(-x)=-shx. \tag{2.9}$$

2) *The function ch is even, i.e.: for every* $x\in\mathbf{R}$,

$$ch(-x)=chx. \tag{2.10}$$

3) *The function th is odd, i.e.: for every* $x\in\mathbf{R}$,

$$th(-x)=-thx. \tag{2.11}$$

4) *The function cth is odd, i.e.: for every* $x\in\mathbf{R}^{*}$,

$$cth(-x)=-cthx. \tag{2.12}$$

5) *The function sch is even, i.e.: for every* $x\in\mathbf{R}$,

$$sch(-x)=schx. \tag{2.13}$$

6) *The function csh is odd, i.e.: for every* $x\in\mathbf{R}^{*}$,

$$csh(-x)=-cshx. \tag{2.14}$$

7) *For every* $x\in\mathbf{R}$,

$$chx\geq 1; \tag{2.15}$$

8) *For every orice* $x\in\mathbf{R}$,

$$thx\in(-1,1). \tag{2.16}$$

***Proof*: 1)** According to the equality (2.1), for every $x\in\mathbf{R}$,

$$\mathrm{sh}(-x)=\frac{e^{-x}-e^{x}}{2}=-\frac{e^{x}-e^{-x}}{2}=-\mathrm{shx};$$

so, the equality (2.9) holds.

2) According to the equality (2.2), for every $x\in\mathbf{R}$,

$$\mathrm{ch}(-x)=\frac{e^{-x}+e^{x}}{2}=\frac{e^{x}+e^{-x}}{2}=\mathrm{chx};$$

so, the equality (2.10) holds.

3) According to the equalities (2.3), (2.9) and (2.10), for every $x\in\mathbf{R}$,

$$\mathrm{th}(-x)=\frac{\mathrm{sh}(-x)}{\mathrm{ch}(-x)}=\frac{-\mathrm{shx}}{\mathrm{chx}}=-\mathrm{thx}.$$

***Otherwise*:** According to the first equality to (2.3′), for every $x\in\mathbf{R}$,

$$\text{th}(-x)=\frac{e^{-x}-e^{x}}{e^{-x}+e^{x}}=-\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\text{th}(-x).$$

Therefore, the equality (2.11) holds.

4) According to the equalities (2.4) and (2.11), for every $x\in\mathbf{R}^*$,

$$\text{cth}(-x)=\frac{1}{\text{th}(-x)}=\frac{1}{-\text{th}x}=-\text{cth}x.$$

***Otherwise*:** According to the equalities (2.4), (2.9) and (2.10), for every $x\in\mathbf{R}^*$,

$$\text{cth}(-x)=\frac{\text{ch}(-x)}{\text{sh}(-x)}=\frac{\text{ch}x}{-\text{sh}x}=-\text{cth}x.$$

***Otherwise*:** According to the first equality to (2.4′), for every $x\in\mathbf{R}^*$,

$$\text{cth}(-x)=\frac{e^{-x}+e^{x}}{e^{-x}-e^{x}}=-\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=\text{cth}(-x).$$

Therefore, the equality (2.12) holds.

5) According to the equalities (2.5) and (2.10), for every $x\in\mathbf{R}$,

$$\text{sch}(-x)=\frac{1}{\text{ch}(-x)}=\frac{1}{\text{ch}x}=\text{sch}x.$$

***Otherwise*:** According to the first equality to (2.5′), for every $x\in\mathbf{R}$,

$$\text{sch}(-x)=\frac{2}{e^{-x}+e^{x}}=\text{sch}x.$$

So, the equality (2.13) holds.

6) According to the equalities (2.6) and (2.9), for every $x\in\mathbf{R}^*$,

$$\text{csh}(-x)=\frac{1}{\text{sh}(-x)}=\frac{1}{-\text{sh}x}=-\text{csh}x.$$

***Otherwise*:** according to the first equality to (2.6′), for every $x\in\mathbf{R}^*$,

$$\text{csh}(-x)=\frac{2}{e^{-x}-e^{x}}=-\frac{2}{e^{x}-e^{-x}}=-\text{csh}x.$$

Therefore, the equality (2.14) holds, also.

7) From the first equality to (2.2′), applying the arithemtic mean – geometric mean inequality, it follows that, for every $x\in\mathbf{R}$, the inequality (2.15) holds.

8) From the first equality to (2.4′) it follows that, for every $x\in\mathbf{R}$,

$$1-\text{th}x=\frac{2\cdot e^{-x}}{e^{x}+e^{-x}}>0 \qquad \text{and} \qquad 1+\text{th}x=\frac{2\cdot e^{x}}{e^{x}+e^{-x}}>0,$$

which shows that, for every $x\in\mathbf{R}$, $\text{th}x\in(-1,1)$. □

3. Fundamental properties of hyperbolic functions

In this section we will present the fundamental properties of hyperbolic functions. The first 38 of these properties, divided into four groups:

A. *„Trigonometric" properties – nine properties;*
B. *The derivatives of hyperbolic functions – six properties;*
C. *The primitives (indefinite integrals) of hyperbolic functions – six properties;*
D. *The monotony and the invertibility of hyperbolic functions – 17 properties*;

they are as follows. All these properties will be proved, some of them in two ways.

Proposition 3.1: *The following statements hold*:

A. „Trigonometric" properties

1) *For every* $x\in\mathbf{R}$,

$$ch^2x-sh^2x=2. \tag{3.1}$$

2) *For every* $x, y\in\mathbf{R}$,

$$sh(x+y)=shx\cdot chy+chx\cdot shy. \tag{3.2}$$

3) *For every* $x, y\in\mathbf{R}$,

$sh(x\text{-}y)=shx\cdot chy\text{-}chx\cdot shy.$ (3.3)

4) *For every* $x, y\in \mathbf{R}$,

$ch(x+y)=chx\cdot chy+shx\cdot shy.$ (3.4)

5) *For every* $x, y\in \mathbf{R}$,

$ch(x\text{-}y)=chx\cdot chy\text{-}shx\cdot shy.$ (3.5)

6) *For every* $x, y\in \mathbf{R}$,

$$th(x+y)=\frac{thx+thy}{1+thx\cdot thy}. \quad (3.6)$$

7) *For every* $x, y\in \mathbf{R}$,

$$th(x\text{-}y)=\frac{thx-thy}{1-thx\cdot thy}. \quad (3.7)$$

8) *For every* $x, y\in \mathbf{R}^*$, *with the property that* $x+y\neq 0$,

$$cth(x+y)=\frac{cthx\cdot cthy+1}{cthx+cthy}. \quad (3.8)$$

9) *For every* $x, y\in \mathbf{R}^*$, *with the property that* $x\neq y$,

$$cth(x\text{-}y)=\frac{cthx\cdot cthy-1}{cthy-cthx}. \quad (3.9)$$

B. The derivatives of hyperbolic functions

10) *For every* $x\in \mathbf{R}$,

$(shx)'=chx.$ (3.10)

11) *For every* $x\in \mathbf{R}$,

$(chx)'=shx.$ (3.11)

12) *For every* $x\in \mathbf{R}$,

$$(thx)'=\frac{1}{ch^2x}=sch^2x. \quad (3.12)$$

13) *For every* $x\in \mathbf{R}^*$,

$$(cthx)'=-\frac{1}{sh^2x}=-csh^2x. \quad (3.13)$$

14) *For every* $x\in \mathbf{R}$,

$$(schx)'=-\frac{shx}{ch^2x}=-thx\cdot schx. \quad (3.14)$$

15) *For every* $x\in \mathbf{R}^*$,

$$(cshx)'=-\frac{chx}{sh^2x}=-cthx\cdot cshx. \quad (3.15)$$

C. The primitives (indefinite integrals) of hyperbolic functions

16) *For every* $x\in \mathbf{R}$,

$$\int shx\cdot dx=chx+C. \quad (3.16)$$

17) *For every* $x\in \mathbf{R}$,

$$\int chx\cdot dx=shx+C. \quad (3.17)$$

18) *For every* $x\in \mathbf{R}$,

$$\int thx\cdot dx=ln(chx)+C. \quad (3.18)$$

19) *For every* $x\in \mathbf{R}^*$,

$$\int cthx\cdot dx=ln|shx|+C. \quad (3.19)$$

20) *For every* $x\in \mathbf{R}$,

$$\int schx\cdot dx=arctg(shx)+C. \quad (3.20)$$

21) *For every* $x\in \mathbf{R}^*$,

$$\int csh x\cdot dx=\frac{1}{2}\cdot ln\left(\frac{chx-1}{chx+1}\right)+C. \tag{3.21}$$

D. The monotony and the invertibility of hyperbolic functions

22) *The function sh is strictly increasing on **R**.*

23) *The function sh is invertible and its inverse is the function*:

$sh^{-1}:\mathbf{R}\to\mathbf{R}$,

where, for every $x\in\mathbf{R}$,

$$sh^{-1}(x)=ln(x+\sqrt{x^2+1}). \tag{3.22}$$

24) *The function ch is strictly decreasing on* $(-\infty,0)$ *and strictly increasing on* $(0,+\infty)$.

25) *The function* ch_1 *- the restriction of function ch to the interval* $(-\infty,0]$, *is invertible and its inverse is the function*:

$ch_1^{-1}:[1,+\infty)\to(-\infty,0)$,

where, for every $x\in[1,+\infty)$,

$$ch_1^{-1}(x)=ln(x-\sqrt{x^2-1}). \tag{3.23}$$

26) *The function* ch_2 *- the restriction of function ch to the interval* $[0,+\infty)$, *is invertible and its inverse is the function*:

$ch_2^{-1}:[1,+\infty)\to[0,+\infty)$,

where, for every $x\in[1,+\infty)$,

$$ch_2^{-1}(x)=ln(x+\sqrt{x^2-1}). \tag{3.24}$$

27) *The function th is strictly increasing on **R**.*

28) *The function th is invertible and its inverse is the function*:

$th^{-1}:(-1,1)\to\mathbf{R}$,

where, for every $x\in(-1,1)$,

$$th^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{1+x}{1-x}\right). \tag{3.25}$$

29) *The function cth is strictly decreasing both on* $(-\infty,0)$, *as well as on* $(0,+\infty)$.

30) *The function* cth_1 *- the restriction of function cth to the interval* $(-\infty,0)$, *is invertible and its inverse is the function*:

$cth_1^{-1}:(-\infty,-1)\to(-\infty,0)$,

where, for every $x\in(-\infty,-1)$,

$$cth_1^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{x+1}{x-1}\right). \tag{3.26}$$

31) *The function* cth_2 *- the restriction of function cth to the interval* $(0,+\infty)$, *is invertible and its inverse is the function*:

$cth_2^{-1}:(1,+\infty)\to(0,+\infty)$,

where, for every $x\in(1,+\infty)$,

$$cth_2^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{x+1}{x-1}\right). \tag{3.26′}$$

32) *The function cth is invertible and its inverse is the function*:

$cth^{-1}:(-\infty,-1)\cup(1,+\infty)\to\mathbf{R}^*$,

where, for every $x\in(-\infty,-1)\cup(1,+\infty)$,

$$cth^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{x+1}{x-1}\right). \tag{3.26″}$$

33) *The function sch is strictly increasing on* $(-\infty,0]$ *and strictly decreasing on* $[0,+\infty)$.

34) *The function* sch_1 *- the restriction of function sch to the interval* $(-\infty,0]$, *is invertible and its inverse is the function*:

$sch_1^{-1}:(0,1]\to(-\infty,0]$,

where, for every $x\in(0,1]$,

$$sch_1^{-1}(x)=ln\left(\frac{1-\sqrt{1-x^2}}{x}\right). \tag{3.27}$$

35) *The function sch_2 - the restriction of function sch to the interval $[0,+\infty)$, is invertible and its inverse is the function*:

$$sch_2^{-1}:(0,1]\to[0,+\infty),$$

where, for every $x\in(0,1]$,

$$sch_2^{-1}(x)=ln\left(\frac{1+\sqrt{1-x^2}}{x}\right). \tag{3.28}$$

36) *The function csh is strictly decreasing both on $(-\infty,0)$, as well as on $(0,+\infty)$.*

37) *The function csh_1 - the restriction of function csh to the interval $(-\infty,0)$ is invertible and its inverse is the function*:

$$csh_1^{-1}:(-\infty,0)\to(-\infty,0),$$

where, for every $x\in(-\infty,0)$,

$$csh_1^{-1}(x)=ln\left(\frac{1-\sqrt{1+x^2}}{x}\right). \tag{3.29}$$

38) *The function csh_2 - the restriction of function csh to the interval $(0,+\infty)$, is invertible and its inverse is the function*:

$$csh_2^{-1}:(0,+\infty)\to(0,+\infty),$$

where, for every $x\in(0,+\infty)$,

$$csh_2^{-1}(x)=ln\left(\frac{1+\sqrt{1+x^2}}{x}\right). \tag{3.30}$$

***Proof*: 1)** According to the equalities (2.1) and (2.2), for every $x\in\mathbf{R}$,

$$ch^2x\text{-}sh^2x=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=\frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4}=1;$$

so, the equality (3.1) holds.

2) According to the equality (2.1), for every $x, y\in\mathbf{R}$,

(1) $sh(x+y)=\dfrac{e^{x+y}-e^{-x-y}}{2}$.

On the other hand, from the equalities (2.1) and (2.2), obtain that:

(2) $$shx\cdot chy+chx\cdot shy=\frac{e^x-e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x+e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$$
$$=\frac{e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}+e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}}{4}=\frac{2\cdot e^{x+y}-2\cdot e^{-x-y}}{4}.$$

From the equalities (1) and (2), follows the equality (3.2).

3) According to the equality (2.1), for every $x, y\in\mathbf{R}$,

(1) $sh(x-y)=\dfrac{e^{x-y}-e^{-x+y}}{2}$.

On the other hand, from the equalities (2.1) şi (2.2), obtain that:

(2) $$shx\cdot chy-chx\cdot shy=\frac{e^x-e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}-\frac{e^x+e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$$
$$=\frac{e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}-e^{x+y}+e^{x-y}-e^{-x+y}+e^{-x-y}}{4}=\frac{2\cdot e^{x-y}-2\cdot e^{-x+y}}{4}.$$

From the equalities (1) and (2), follows the equality (3.3).

4) According to the equality (2.2), for every x, y$\in$**R**,

(1) $\mathrm{ch}(x+y)=\frac{e^{x+y}+e^{-x-y}}{2}$.

On the other hand, from the equalities (2.1) and (2.2), obtain that:

(2) $\mathrm{ch}x\cdot\mathrm{ch}y+\mathrm{sh}x\cdot\mathrm{sh}y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}+e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y}}{4}=\frac{2\cdot e^{x+y}+2\cdot e^{-x-y}}{4}.$$

From the equalities (1) and (2), follows the equality (3.4).

5) According to the equality (2.2), for every x, y$\in$**R**,

(1) $\mathrm{ch}(x-y)=\frac{e^{x-y}+e^{-x+y}}{2}$.

On the other hand, from the equalities (2.1) and (2.2), obtain that:

(2) $\mathrm{ch}x\cdot\mathrm{ch}y-\mathrm{sh}x\cdot\mathrm{sh}y=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}-\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$

$$=\frac{e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y}-e^{x+y}+e^{x-y}+e^{-x+y}-e^{-x-y}}{4}=\frac{2\cdot e^{x+y}+2\cdot e^{-x-y}}{4}.$$

From the equalities (1) and (2), follows the equality (3.5).

6) According to the equalities (2.3), (3.2) and (3.4), for every x, y$\in$**R**,

$$\mathrm{th}(x+y)=\frac{\mathrm{sh}(x+y)}{\mathrm{ch}(x+y)}=\frac{\mathrm{sh}x\cdot\mathrm{ch}y+\mathrm{ch}x\cdot\mathrm{sh}y}{\mathrm{ch}x\cdot\mathrm{ch}y+\mathrm{sh}x\cdot\mathrm{sh}y}=\frac{\frac{\mathrm{sh}x}{\mathrm{ch}x}+\frac{\mathrm{sh}y}{\mathrm{ch}y}}{1+\frac{\mathrm{sh}x}{\mathrm{ch}x}\cdot\frac{\mathrm{sh}y}{\mathrm{ch}y}}=\frac{\mathrm{th}x+\mathrm{th}y}{1+\mathrm{th}x\cdot\mathrm{th}y}.$$

So, we obtained the equality (3.6). Of course that if x, y$\in$**R**, because thx, thy$\in$(-1,1) – according to the relation (2.16), it follows that: $1+\mathrm{th}x\cdot\mathrm{th}y\neq0$.

7) According to the equalities (2.3), (3.3) and (3.5), for every x, y$\in$**R**,

$$\mathrm{th}(x-y)=\frac{\mathrm{sh}(x-y)}{\mathrm{ch}(x-y)}=\frac{\mathrm{sh}x\cdot\mathrm{ch}y-\mathrm{ch}x\cdot\mathrm{sh}y}{\mathrm{ch}x\cdot\mathrm{ch}y-\mathrm{sh}x\cdot\mathrm{sh}y}=\frac{\frac{\mathrm{sh}x}{\mathrm{ch}x}-\frac{\mathrm{sh}y}{\mathrm{ch}y}}{1-\frac{\mathrm{sh}x}{\mathrm{ch}x}\cdot\frac{\mathrm{sh}y}{\mathrm{ch}y}}=\frac{\mathrm{th}x-\mathrm{th}y}{1-\mathrm{th}x\cdot\mathrm{th}y}.$$

So, we obtained the equality (3.7). Of course that if x, y$\in$**R**, because thx, thy$\in$(-1,1) – according to the relation (2.16), it follows that: $1-\mathrm{th}x\cdot\mathrm{th}y\neq0$.

8) According to the equalities (2.4), (3.4) and (3.2), for every x, y$\in$**R***, with the property that $x+y\neq0$,

$$\mathrm{cth}(x+y)=\frac{\mathrm{ch}(x+y)}{\mathrm{sh}(x+y)}=\frac{\mathrm{ch}x\cdot\mathrm{ch}y+\mathrm{sh}x\cdot\mathrm{sh}y}{\mathrm{sh}x\cdot\mathrm{ch}y+\mathrm{ch}x\cdot\mathrm{sh}y}=\frac{\frac{\mathrm{ch}x}{\mathrm{sh}x}\cdot\frac{\mathrm{ch}y}{\mathrm{sh}y}+1}{\frac{\mathrm{ch}x}{\mathrm{sh}x}+\frac{\mathrm{ch}y}{\mathrm{sh}y}}=\frac{\mathrm{cth}x\cdot\mathrm{cth}y+1}{\mathrm{cth}x+\mathrm{cth}y}.$$

So, we obtained the equality (3.8). Of course that, x, y$\in$**R***, so that there cthx, respectively cthy. On the other hand, should that: $\mathrm{cth}x+\mathrm{cth}y\neq0$, i.e. we have the following equivalences:

$$\frac{e^x+e^{-x}}{e^x-e^{-x}}+\frac{e^y+e^{-y}}{e^y-e^{-y}}\neq0\Leftrightarrow\frac{e^{2x}+1}{e^{2x}-1}+\frac{e^{2y}+1}{e^{2y}-1}\neq0\Leftrightarrow\frac{e^{2x+2y}-e^{2x}+e^{2y}-1+e^{2x+2y}+e^{2x}-e^{2y}-1}{(e^{2x}-1)\cdot(e^{2y}-1)}\neq0$$

$$\Leftrightarrow\frac{2\cdot(e^{2x+2y}-1)}{(e^{2x}-1)\cdot(e^{2y}-1)}\neq0\Leftrightarrow x+y\neq0.$$

So, the conditions from statement are correct.

9) According to the equalities (2.4), (3.5) and (3.3), for every x, y$\in$**R***, with the property that $x\neq y$,

$$\mathrm{cth}(x\text{-}y)=\frac{\mathrm{ch}(x-y)}{\mathrm{sh}(x-y)}=\frac{\mathrm{ch}x\cdot\mathrm{ch}y-\mathrm{sh}x\cdot\mathrm{sh}y}{\mathrm{sh}x\cdot\mathrm{ch}y-\mathrm{ch}x\cdot\mathrm{sh}y}=\frac{\frac{\mathrm{ch}x}{\mathrm{sh}x}\cdot\frac{\mathrm{ch}y}{\mathrm{sh}y}-1}{\frac{\mathrm{ch}y}{\mathrm{sh}y}-\frac{\mathrm{ch}x}{\mathrm{sh}x}}=\frac{\mathrm{cth}x\cdot\mathrm{cth}y-1}{\mathrm{cth}y-\mathrm{cth}x}.$$

So, we obtained the equality (3.9). Because, $\mathrm{cth}x-\mathrm{cth}y\neq 0$, we have the following equivalences:

$$\frac{e^x+e^{-x}}{e^x-e^{-x}}-\frac{e^y+e^{-y}}{e^y-e^{-y}}\neq 0\Leftrightarrow\frac{e^{2x}+1}{e^{2x}-1}-\frac{e^{2y}+1}{e^{2y}-1}\neq 0\Leftrightarrow e^{2x+2y}-e^{2x}+e^{2y}-1-e^{2x+2y}-e^{2x}+e^{2y}+1\neq 0$$

$$\Leftrightarrow 2\cdot(e^{2x}-e^{2y})\neq 0\Leftrightarrow x\neq y.$$

Therefore, the conditions from statement are correct.

10) According to the equality (2.1), for every $x\in\mathbf{R}$,

$$(\mathrm{sh}x)'=\left(\frac{e^x-e^{-x}}{2}\right)'=\frac{e^x+e^{-x}}{2}=\mathrm{ch}x;$$

so, the equality (3.10) holds.

11) According to the equality (2.2), for every $x\in\mathbf{R}$,

$$(\mathrm{ch}x)'=\left(\frac{e^x+e^{-x}}{2}\right)'=\frac{e^x-e^{-x}}{2}=\mathrm{sh}x;$$

so, the equality (3.11) holds.

12) According to the equalities (2.3), (3.10), (3.11) and (2.5), for every $x\in\mathbf{R}$,

$$(\mathrm{th}x)'=\left(\frac{\mathrm{sh}x}{\mathrm{ch}x}\right)'=\frac{(\mathrm{sh}x)'\cdot\mathrm{ch}x-\mathrm{sh}x\cdot(\mathrm{ch}x)'}{\mathrm{ch}^2x}=\frac{\mathrm{ch}^2x-\mathrm{sh}^2x}{\mathrm{ch}^2x}=\frac{1}{\mathrm{ch}^2x}=\mathrm{sch}^2x;$$

***or otherwise*:** according to the equalities (2.3), (2.2) and (2.5), for every $x\in\mathbf{R}$,

$$(\mathrm{th}x)'=\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)'=\frac{(e^x-e^{-x})'\cdot(e^x+e^{-x})-(e^x-e^{-x})\cdot(e^x+e^{-x})'}{(e^x+e^{-x})^2}$$

$$=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}=\frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{(e^x+e^{-x})^2}=\frac{4}{(e^x+e^{-x})^2}=\frac{1}{\mathrm{ch}^2x}=\mathrm{sch}^2x;$$

so, the equality (3.12) holds.

13) According to the equalities (2.4), (3.11), (3.10) and (2.6), for every $x\in\mathbf{R}^*$,

$$(\mathrm{cth}x)'=\left(\frac{\mathrm{ch}x}{\mathrm{sh}x}\right)'=\frac{(\mathrm{ch}x)'\cdot\mathrm{sh}x-\mathrm{ch}x\cdot(\mathrm{sh}x)'}{\mathrm{sh}^2x}=\frac{\mathrm{sh}^2x-\mathrm{ch}^2x}{\mathrm{sh}^2x}=-\frac{1}{\mathrm{sh}^2x}=-\mathrm{csh}^2x;$$

***or otherwise*:** according to the equalities (2.4) and (2.6), for every $x\in\mathbf{R}^*$,

$$(\mathrm{cth}x)'=\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)'=\frac{(e^x+e^{-x})'\cdot(e^x-e^{-x})-(e^x+e^{-x})\cdot(e^x-e^{-x})'}{(e^x-e^{-x})^2}$$

$$=\frac{(e^x-e^{-x})^2-(e^x+e^{-x})^2}{(e^x-e^{-x})^2}=\frac{e^{2x}-2+e^{-2x}-e^{2x}-2-e^{-2x}}{(e^x-e^{-x})^2}=-\frac{4}{(e^x-e^{-x})^2}=-\frac{1}{\mathrm{sh}^2x}=-\mathrm{csh}^2x;$$

***or otherwise*:** according to the equalities (2.4), (3.12) and (2.6), for every $x\in\mathbf{R}^*$,

$$(\mathrm{cth}x)'=\left(\frac{1}{\mathrm{th}x}\right)'=-\frac{(\mathrm{th}x)'}{\mathrm{th}^2x}=-\frac{\frac{1}{\mathrm{ch}^2x}}{\mathrm{th}^2x}=-\frac{\frac{1}{\mathrm{ch}^2x}}{\frac{\mathrm{sh}^2x}{\mathrm{ch}^2x}}=-\frac{1}{\mathrm{sh}^2x}=-\mathrm{csh}^2x;$$

therefore, the equality (3.13) holds.

14) According to the equalities (2.5), (3.11) and (2.3), for every $x\in\mathbf{R}$,

$$(\mathrm{sch}x)'=\left(\frac{1}{\mathrm{ch}x}\right)'=-\frac{\mathrm{sh}x}{\mathrm{ch}^2x}=-\frac{\mathrm{sh}x}{\mathrm{ch}x}\cdot\frac{1}{\mathrm{ch}x}=-\mathrm{th}x\cdot\mathrm{sch}x;$$

therefore, the equality (3.14) holds.

15) According to the equalities (2.6), (3.10) and (2.4), for every $x\in\mathbf{R}^*$,

$$(\mathrm{cshx})'=\left(\frac{1}{\mathrm{shx}}\right)'=-\frac{\mathrm{chx}}{\mathrm{sh}^2\mathrm{x}}=-\frac{\mathrm{chx}}{\mathrm{shx}}\cdot\frac{1}{\mathrm{shx}}=-\mathrm{cthx}\cdot\mathrm{cshx};$$

therefore, the equality (3.15) holds.

16) According to the equalities (2.1) and (2.2), for every $x\in\mathbf{R}$,

$$\int \mathrm{shx}\cdot dx=\int\frac{e^x-e^{-x}}{2}\cdot dx=\frac{e^x+e^{-x}}{2}=\mathrm{chx}+C;$$

so, the equality (3.16) holds.

***Otherwise*:** The equality from statement follows from the equality (3.11), by passing to the integral.

17) According to the equalities (2.2) and (2.1), for every $x\in\mathbf{R}$,

$$\int \mathrm{chx}\cdot dx=\int\frac{e^x+e^{-x}}{2}\cdot dx=\frac{e^x-e^{-x}}{2}=\mathrm{shx}+C;$$

so, the equality (3.17) holds.

***Otherwise*:** The equality from statement follows from the equality (3.10), by passing to the integral.

18) According to the equalities (2.3) and (2.2), for every $x\in\mathbf{R}$,

$$\int \mathrm{thx}\cdot dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}\cdot dx=\int\frac{e^{2x}-1}{e^{2x}+1}\cdot dx=\int\frac{e^{2x}-1}{e^x\cdot(e^{2x}+1)}\cdot e^x\cdot dx.$$

By the substitution from the variable:

$e^x=y$,

obtain that:

$e^x\cdot dx=dy$

and we compute the obtained primitive (indefinite integral):

$$\int\frac{y^2-1}{y\cdot(y^2+1)}\cdot dy=\int\left(\frac{2\cdot y}{y^2+1}-\frac{1}{y}\right)\cdot dy=\int\frac{2\cdot y}{y^2+1}\cdot dy-\int\frac{1}{y}\cdot dy=\ln(y^2+1)-\ln y+C=\ln\left(\frac{y^2+1}{y}\right)+C.$$

Of course that, here we made the following decomposition in simple fractions:

$$\frac{y^2-1}{y\cdot(y^2+1)}=\frac{A}{y}+\frac{B\cdot y+C}{y^2+1},$$

where:

$$A=\left.\frac{y^2-1}{y^2+1}\right|_{y=0}=-1 \quad\text{and}\quad \frac{B\cdot y+C}{y^2+1}=\frac{y^2-1}{y\cdot(y^2+1)}-\frac{1}{y}=\frac{2\cdot y}{y^2+1};$$

whence it follows that:

$$\frac{y^2-1}{y\cdot(y^2+1)}=\frac{2\cdot y}{y^2+1}-\frac{1}{y}.$$

Returning to the original primitive obtain that:

$$\int \mathrm{thx}\cdot dx=\ln\left(\frac{e^{2x}+1}{e^x}\right)+C=\ln\left(\frac{e^x+e^{-x}}{2}\right)+\ln 2+C=\ln(\mathrm{chx})+C.$$

***Otherwise*:** According to the equalities (2.3) and (3.11), for every $x\in\mathbf{R}$,

$$\int \mathrm{thx}\cdot dx=\int\frac{\mathrm{shx}}{\mathrm{chx}}\cdot dx=\int\frac{(\mathrm{chx})'}{\mathrm{chx}}\cdot dx=\ln(\mathrm{chx})+C.$$

So, we can say that the equality (3.18) holds.

19) According to the equalities (2.4) and (2.1), for every $x\in\mathbf{R}^*$,

$$\int \mathrm{cthx}\cdot dx=\int\frac{e^x+e^{-x}}{e^x-e^{-x}}\cdot dx=\int\frac{e^{2x}+1}{e^{2x}-1}\cdot dx=\int\frac{e^{2x}+1}{e^x\cdot(e^{2x}-1)}\cdot e^x\cdot dx$$

By the substitution from the variable:

$e^x=y$,

obtain that:

$e^x\cdot dx=dy$

and we compute the obtained primitive:

$$\int \frac{y^2+1}{y\cdot(y^2-1)}\cdot dy = \int\left(-\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y+1}\right)\cdot dy = -\int\frac{1}{y}\cdot dy + \int\frac{1}{y-1}\cdot dy + \int\frac{1}{y+1}\cdot dy$$

$$= -\ln y+\ln|y-1|+\ln(y+1)+C=\ln\left|\frac{y^2-1}{y}\right|+C.$$

Returning to the original primitive obtain that:

$$\int \mathrm{cthx}\cdot dx = \ln\left|\frac{e^{2x}-1}{e^x}\right|+C=\ln\left|\frac{e^x-e^{-x}}{2}\right|+\ln 2+C=\ln|\mathrm{shx}|+C.$$

***Otherwise*:** According to the equalities (2.4) and (3.10), for every $x\in\mathbf{R}^*$,

$$\int \mathrm{cthx}\cdot dx = \int\frac{\mathrm{chx}}{\mathrm{shx}}\cdot dx = \int\frac{(\mathrm{shx})'}{\mathrm{shx}}\cdot dx = \ln(\mathrm{chx})+C.$$

So, we can say that the equality (3.19) holds.

20) According to the equalities (2.5), (3.1) and (3.10), for every $x\in\mathbf{R}$,

$$\int \mathrm{schx}\cdot dx = \int\frac{1}{\mathrm{chx}}\cdot dx = \int\frac{\mathrm{chx}}{\mathrm{ch}^2x}\cdot dx = \int\frac{(\mathrm{shx})'}{1+\mathrm{sh}^2x}\cdot dx = \mathrm{arctg}(\mathrm{shx})+C;$$

so, the equality (3.20) holds.

21) According to the equalities (2.6), (3.1) and (3.11), for every $x\in\mathbf{R}^*$,

(1) $$\int \mathrm{cshx}\cdot dx = \int\frac{1}{\mathrm{shx}}\cdot dx = \int\frac{\mathrm{shx}}{\mathrm{sh}^2x}\cdot dx = \int\frac{(\mathrm{chx})'}{\mathrm{ch}^2x-1}\cdot dx\ .$$

Making the change of variable:

(2) chx=y,

obtain that:

shx·dx=dy

and we have to compute:

(3) $$\int\frac{1}{y^2-1}\cdot dy = \frac{1}{2}\cdot\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)\cdot dy = \frac{1}{2}\cdot\int\frac{1}{y-1}\cdot dy - \frac{1}{2}\cdot\int\frac{1}{y+1}\cdot dy$$

$$= \frac{1}{2}\cdot\ln(y-1)-\frac{1}{2}\cdot\ln(y+1)+C = \frac{1}{2}\cdot\ln\frac{y-1}{y+1}+C.$$

Then, from the equalities (1), (2) and (3), follows the equality (3.21).

22) Indeed, because, according to the equalities (3.10) and (2.2), for every $x\in\mathbf{R}$,

$$(\mathrm{shx})' = \frac{e^x+e^{-x}}{2} > 0,$$

it follows that the function sh is strictly increasing on **R**.

***Otherwise*:** Let be x, $y\in\mathbf{R}$, such that $x<y$. Then $e^x<e^y$ and:

$$\mathrm{shx}-\mathrm{shy} = \frac{e^x-e^{-x}}{2} - \frac{e^y-e^{-y}}{2} = \frac{e^{2x}-1}{2\cdot e^x} - \frac{e^{2y}-1}{2\cdot e^y} = \frac{e^{2x+y}-e^y-e^{x+2y}+e^x}{2\cdot e^{x+y}}$$

$$= \frac{e^{x+y}\cdot(e^x-e^y)+(e^x-e^y)}{2\cdot e^{x+y}} = \frac{(e^x-e^y)\cdot(e^{x+y}+1)}{2\cdot e^{x+y}} < 0,$$

that this shx<shy, which shows that the function sh is strictly increasing on **R**.

23) First we observe that the function sh is continuous on **R**, because is a sum, respectively a fraction from two continuous functions, on **R**. So, the function sh it has the Darboux property on **R**. On the other hand,

$$\lim_{x\to-\infty}\mathrm{shx} = \lim_{x\to-\infty}\frac{e^x-e^{-x}}{2} = -\infty \quad \text{and} \quad \lim_{x\to+\infty}\mathrm{shx} = \lim_{x\to+\infty}\frac{e^x-e^{-x}}{2} = +\infty,$$

which, together with the Darboux property, implies that the function sh is surjective. Then, because this function is strictly increasing on **R**, it follows that she is injective. In conclusion, the function sh is bijective and, thus, it is and invertible.

***Otherwise*:** According to the equality (2.1), for every x, $y\in\mathbf{R}$,

$$\text{shx=shy} \Leftrightarrow \frac{e^x - e^{-x}}{2} = \frac{e^y - e^{-y}}{2} \Leftrightarrow \frac{e^{2x} - 1}{2 \cdot e^x} = \frac{e^{2y} - 1}{2 \cdot e^y} \Leftrightarrow e^{2x+y}\text{-}e^y = e^{x+2y}\text{-}e^x$$

$$\Leftrightarrow e^{2x+y}-e^{x+2y}-e^y+e^x=0 \Leftrightarrow e^{x+y}\cdot(e^x-e^y)+(e^x-e^y)=0 \Leftrightarrow (e^x-e^y)\cdot(e^{x+y}+1)=0 \Leftrightarrow x=y,$$

which shows that the function sh is injective. The surjectivity's sh we can deduce and such: consider a certain element $y\in\mathbf{R}$ and solving the equation:

(1) $\text{shx}=y$,

i.e., according to the equality (2.1):

$$\frac{e^x - e^{-x}}{2} = y.$$

This equation becomes:

$$e^{2x}-2\cdot y\cdot e^x-1=0,$$

whence it follows that:

$$e^x=y+\sqrt{y^2+1}>0,$$

and, thus,

(2) $x=\ln(y+\sqrt{y^2+1})\in\mathbf{R}$.

Therefore, for every $y\in\mathbf{R}$, the equation (1) has a real solution; so, the function sh is surjective. From equality (2) it follows that the inverse this function is the function:

$$\text{sh}^{-1}x : \mathbf{R} \to \mathbf{R},$$

given from law: for every $x\in\mathbf{R}$,

$$\text{sh}^{-1}(x)=\ln(x+\sqrt{x^2+1}).$$

24) Indeed, because, according to the equalities (3.11) and (2.1), for every $x\in\mathbf{R}$,

$$(\text{chx})' = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2 \cdot e^x},$$

it follows that, for every $x\in(-\infty,0)$, $(\text{chx})'<0$ and, for every $x\in(0,+\infty)$, $(\text{chx})'>0$; which shows that the function ch is strictly decreasing on $(-\infty,0)$ and strictly increasing on $(0,+\infty)$.

***Otherwise*:** According to the equality (2.2), for every $x, y\in\mathbf{R}$,

$$\text{chx-chy} = \frac{e^x + e^{-x}}{2} - \frac{e^y + e^{-y}}{2} = \frac{e^{2x} + 1}{2 \cdot e^x} - \frac{e^{2y} + 1}{2 \cdot e^y} = \frac{e^{2x+y} + e^y - e^{x+2y} - e^x}{2 \cdot e^{x+y}}$$

$$= \frac{e^{x+y} \cdot (e^x - e^y) - (e^x - e^y)}{2 \cdot e^{x+y}} = \frac{(e^x - e^y) \cdot (e^{x+y} - 1)}{2 \cdot e^{x+y}}.$$

Now, if $x, y\in(-\infty,0)$, such that $x<y$, then $e^x<e^y$, $e^{x+y}<1$ and, thus, chx>chy, which shows that the function ch is strictly decreasing on $(-\infty,0)$, and, if $x, y\in(0,+\infty)$, such that $x<y$, then $e^x<e^y$, $e^{x+y}>1$ and, thus, chx<chy, which shows that the function ch is strictly increasing on $(0,+\infty)$.

25) First we observe that the function:

$$\text{ch}_1 : (-\infty,0] \to [1,\infty),$$

given from law: for every $x\in(-\infty,0]$,

$$\text{ch}_1(x)=\text{chx},$$

is continuous on the interval $(-\infty,0]$, because it is a sum, respectively a fraction o two continuous functions, on this interval, according to the equality (2.2). So, the function ch_1 has the Darboux property on the interval $(-\infty,0]$. On the other hand,

$$\lim_{x\to-\infty} \text{ch}_1x = \lim_{x\to-\infty} \frac{e^x + e^{-x}}{2} = +\infty \quad \text{and} \quad \lim_{\substack{x\to 0 \\ x<0}} \text{ch}_1x = \lim_{\substack{x\to 0 \\ x<0}} \frac{e^x + e^{-x}}{2} = 1,$$

which, together with the Darboux property, implies that the function ch_1 is surjective. Then, because this function is continuous and strictly decreasing on the interval $(-\infty,0]$, it follows that it is injective. In conclusion, the function ch_1 is bijective and, thus, it is invertible.

***Otherwise*:** According to the equality (2.2), for every $x, y\in(-\infty,0]$,

$$\text{ch}_1x=\text{ch}_1y \Leftrightarrow \frac{e^x + e^{-x}}{2} = \frac{e^y + e^{-y}}{2} \Leftrightarrow \frac{e^{2x} + 1}{2 \cdot e^x} = \frac{e^{2y} + 1}{2 \cdot e^y} \Leftrightarrow e^{2x+y}+e^y=e^{x+2y}+e^x$$

$$\Leftrightarrow e^{2x+y}\text{-}e^{x+2y}+e^{y}\text{-}e^{x}=0 \Leftrightarrow e^{x+y}\cdot(e^{x}\text{-}e^{y})\text{-}(e^{x}\text{-}e^{y})=0 \Leftrightarrow (e^{x}\text{-}e^{y})\cdot(e^{x+y}\text{-}1)=0 \Leftrightarrow x=y,$$

which shows that the function ch_1 is injective. The surjectivity of ch_1 we can deduce and such: consider a certain element $y\in[1,+\infty)$ and solve the equation:

(1) $ch_1x=y$,

i.e., according to the equality (2.2):

$$\frac{e^{x}+e^{-x}}{2}=y.$$

This equation becomes:

$$e^{2x}\text{-}2\cdot y\cdot e^{x}+1=0,$$

whence it follows that:

$$e^{x}=y\text{-}\sqrt{y^{2}-1}\in(0,1],$$

and, thus,

(2) $x=\ln(y\text{-}\sqrt{y^{2}-1})\in(\text{-}\infty,0]$.

Therefore, for every $y\in[1,+\infty)$, the equation (1) has a solution in the interval $(\text{-}\infty,0]$; so, the function ch_1 is surjective. From the equality (2) it follows that inverse of this function is the function:

$$ch_1^{-1} : [1,+\infty) \to (\text{-}\infty,0],$$

where, for every $x\in[1,+\infty)$,

$$ch_1^{-1}(x)=\ln(x\text{-}\sqrt{x^{2}-1}).$$

26) First we observe that the function:

$$ch_2 : [0,+\infty) \to [1,\infty),$$

given from law: for every $x\in[0,+\infty)$,

$$ch_2(x)=chx,$$

is continuous on the interval $[0,+\infty)$, because is a sum, respectively a fraction o two continuous functions, on this interval, according to the equality (2.2). So, the function ch_2 has the Darboux property on the interval $[0,+\infty)$. On the other hand,

$$\lim_{\substack{x\to0\\x>0}} ch_2x=\lim_{\substack{x\to0\\x>0}}\frac{e^{x}+e^{-x}}{2}=1 \quad \text{and} \quad \lim_{x\to+\infty} ch_2x=\lim_{x\to\infty}\frac{e^{x}+e^{-x}}{2}=+\infty,$$

which, together with the Darboux property, implies that the function ch_2 is surjective. Then, because this function is continuous and strictly increasing on the interval $[0,+\infty)$, it follows that it is injective. In conclusion, the function ch_2 is bijective and, thus, it is invertible.

***Otherwise*:** According to the equality (2.2), for every $x, y\in[0,+\infty)$, the following equivalences hold:

$$ch_2x=ch_2y \Leftrightarrow \frac{e^{x}+e^{-x}}{2}=\frac{e^{y}+e^{-y}}{2} \Leftrightarrow \frac{e^{2x}+1}{2\cdot e^{x}}=\frac{e^{2y}+1}{2\cdot e^{y}} \Leftrightarrow e^{2x+y}+e^{y}=e^{x+2y}+e^{x}$$

$$\Leftrightarrow e^{2x+y}\text{-}e^{x+2y}+e^{y}\text{-}e^{x}=0 \Leftrightarrow e^{x+y}\cdot(e^{x}\text{-}e^{y})\text{-}(e^{x}\text{-}e^{y})=0 \Leftrightarrow (e^{x}\text{-}e^{y})\cdot(e^{x+y}\text{-}1)=0 \Leftrightarrow x=y,$$

because $e^{x+y}>1$, which shows that the function ch_2 is injective. The surjectivity of ch_2 we can deduce and such: consider a certain element $y\in[1,+\infty)$ and solve the equation:

(1) $ch_2x=y$,

where, according to the equality (2.2):

$$\frac{e^{x}+e^{-x}}{2}=y.$$

This equation becomes:

$$e^{2x}\text{-}2\cdot y\cdot e^{x}+1=0,$$

whence it follows that:

$$e^{x}=y+\sqrt{y^{2}-1}\in[1,+\infty),$$

and, thus,

(2) $x=\ln(y+\sqrt{y^{2}-1})\in[0,+\infty)$.

Therefore, for every $y\in[1,+\infty)$, the equation (1) has a solution in the interval $[0,+\infty)$; so, the function ch_2 is surjective. From the equality (2) it follows that inverse of this function is the function:

$ch_2^{-1}: [1,+\infty) \to [0,+\infty)$,

where, for every $x\in[1,+\infty)$,

$ch_2^{-1}(x)=\ln(x+\sqrt{x^2-1})$.

27) Indeed, because, according to the equalities (3.12) and (2.3′), for every $x\in\mathbf{R}$,

$$(thx)'=\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2>0,$$

it follows that the function th is strictly increasing on **R**.

***Otherwise*:** According to the first equality from (2.3′), let be x, $y\in\mathbf{R}$, such that x<y. Then $e^x<e^y$ and

$$thx\text{-}thy=\frac{e^x-e^{-x}}{e^x+e^{-x}}-\frac{e^y-e^{-y}}{e^y+e^y}=\frac{e^{2x}-1}{e^{2x}+1}-\frac{e^{2y}-1}{e^{2y}+1}=\frac{e^{2x+2y}-e^{2y}-e^{2x+2y}+e^{2x}}{(e^{2x}+1)\cdot(e^{2y}+1)}$$

$$=\frac{e^{2x}-e^{2y}}{(e^{2x}+1)\cdot(e^{2y}+1)}=\frac{(e^x-e^y)\cdot(e^x+e^y)}{(e^{2x}+1)\cdot(e^{2y}+1)}<0,$$

i.e. thx<thy, which shows that the function th is strictly increasing on **R**.

28) First we observe that the function th is continuous on **R**, because is a sum, respectively a fraction o two continuous functions, on **R** – see the equalities (2.3′). So, the function th has the Darboux property on **R**. On the other hand,

$$\lim_{x\to-\infty} thx=\lim_{x\to-\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=-1 \quad \text{and} \quad \lim_{x\to+\infty} thx=\lim_{x\to+\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=1,$$

which, together with the Darboux property, implies that the function th is surjective. Then, because this function is continuous and strictly increasing on **R**, it follows that it is injective. In conclusion, the function th is bijective and, thus, it is invertible.

***Otherwise*:** According to the first equality from (2.3′), for x, $y\in\mathbf{R}$, the following equivalences hold:

$$thx=thy \Leftrightarrow \frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{e^y-e^{-y}}{e^y+e^{-y}} \Leftrightarrow \frac{e^{2x}-1}{e^{2x}+1}=\frac{e^{2y}-1}{e^{2y}+1}$$

$$\Leftrightarrow e^{2x+2y}-e^{2y}+e^{2x}-1=e^{2x+2y}-e^{2x}+e^{2y}-1 \Leftrightarrow e^{2x}=e^{2x} \Leftrightarrow x=y,$$

which shows that the function th is injective. The surjectivity of th we can deduce and such: consider a certain element $y\in(-1,1)$ and solve the equation:

(1) thx=y,

i.e., according to the first equality from (2.3′):

$$\frac{e^x-e^{-x}}{e^x+e^{-x}}=y.$$

This equation becomes:

$$e^{2x}=\frac{1+y}{1-y}\in(0,+\infty),$$

whence it follows that:

(2) $x=\frac{1}{2}\cdot\ln\frac{1+y}{1-y}\in\mathbf{R}$.

Therefore, for every $y\in\mathbf{R}$, the equation (1) has a real solution; so, the function th is surjective. From the equality (2) it follows that inverse of this function is the function:

$th^{-1}: (-1,1) \to \mathbf{R}$,

where, for every $x\in(-1,1)$,

$$th^{-1}x=\frac{1}{2}\cdot\ln\frac{1+x}{1-x}.$$

29) Indeed, because, according to the equalities (3.13) and (2.4′), for every $x\in\mathbf{R}^*$,

$$(cthx)'=-\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2<0,$$

it follows that the function cth is strictly decreasing both on $(-\infty,0)$, and the $(0,+\infty)$.

Otherwise: According to the first equality from (2.4′), for every x, y$\in$**R***, we have the equalities:

$$\text{cthx-cthy}=\frac{e^x+e^{-x}}{e^x-e^{-x}}-\frac{e^y+e^{-y}}{e^y-e^y}=\frac{e^{2x}+1}{e^{2x}-1}-\frac{e^{2y}+1}{e^{2y}-1}=\frac{e^{2x+2y}+e^{2y}-e^{2x+2y}-e^{2x}}{(e^{2x}-1)\cdot(e^{2y}-1)}$$

$$=-\frac{e^{2x}-e^{2y}}{(e^{2x}-1)\cdot(e^{2y}-1)}=-\frac{(e^x-e^y)\cdot(e^x+e^y)}{(e^{2x}-1)\cdot(e^{2y}-1)}.$$

Now, if x, y$\in$(-∞,0), such that x<y, then $e^x<e^y$, $e^{2x}<1$, $e^{2y}<1$ and, thus, cthx>cthy, which shows that the function cth is strictly decreasing on (-∞,0), and if x, y$\in$(0,+∞), such that x<y, then $e^x<e^y$, $e^{2x}>1$, $e^{2y}>1$ and, thus, chx<chy, which shows that the function ch is (also) strictly decreasing on (0,+∞).

30) First we observe that the function:

$\text{cth}_1 : (-\infty,0) \to (-\infty,-1)$,

given from law: for every x$\in$(-∞,0),

$\text{cth}_1(x)=\text{cthx}$,

is continuous on the interval (-∞,0), because is a sum, respectively a fraction o two continuous functions, on this interval, according to the equalities (2.4′). So, the function cth_1 has the Darboux property on the interval (-∞,0). On the other hand,

$$\lim_{x\to-\infty}\text{cth}_1x=\lim_{x\to-\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}=-1 \quad \text{and} \quad \lim_{\substack{x\to0\\x<0}}\text{cth}_1x=\lim_{\substack{x\to0\\x<0}}\frac{e^x+e^{-x}}{e^x-e^{-x}}=-\infty,$$

which, together with the Darboux property, implies that the function cth_1 is surjective. Then, because this function is continuous and strictly decreasing on the interval (-∞,0), it follows that it is injective. In conclusion, the function cth_1 is bijective and, thus, it is invertible.

Otherwise: According to the first equality from (2.4′), for x, y$\in$(-∞,0), we have the equivalences:

$$\text{cth}_1x=\text{cth}_1y \Leftrightarrow \frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{e^y+e^{-y}}{e^y-e^{-y}} \Leftrightarrow \frac{e^{2x}+1}{e^{2x}-1}=\frac{e^{2y}+1}{e^{2y}-1}$$

$$\Leftrightarrow e^{2x+y}+e^{2y}-e^{2x}-1=e^{x+2y}+e^{2x}-e^{2y}-1 \Leftrightarrow e^{2x}=e^{2y} \Leftrightarrow x=y,$$

which shows that the function cth_1 is injective. The surjectivity of cth_1 we can deduce and such: consider a certain element y$\in$(-∞,-1) and solve the equation:

(1) $\text{cth}_1x=y$,

i.e., according to the first equality from (1,4′):

$$\frac{e^x+e^{-x}}{e^x-e^{-x}}=y.$$

This equation becomes:

$$e^{2x}=\frac{y+1}{y-1}\in(0,1),$$

and, thus,

(2) $x=\frac{1}{2}\cdot\ln\frac{y+1}{y-1}\in(-\infty,0)$.

Therefore, for every y$\in$(-∞,-1), the equation (1) has a solution in the interval (-∞,0); so, the function cth_1 is surjective. From the equality (2) it follows that inverse of this function is the function:

$\text{cth}_1^{-1} : (-\infty,-1) \to (-\infty,0)$,

where, for every x$\in$(-∞,-1),

$$\text{cth}_1^{-1}(x)=\frac{1}{2}\cdot\ln\frac{x+1}{x-1}.$$

31) First we observe that the function:

$\text{cth}_2 : (0,+\infty) \to (1,+\infty)$,

given from law: for every x$\in$(0,+∞),

$\text{cth}_2(x)=\text{cthx}$,

is continuous on the interval (0,+∞), because is a sum, respectively a fraction o two continuous functions, on this interval, according to the equalities (1,4′). So, the function cth_2 has the Darboux property on the interval $(0,+\infty)$. On the other hand,

$$\lim_{\substack{x\to 0\\ x>0}} \mathrm{cth}_2 x=\lim_{\substack{x\to 0\\ x>0}} \frac{e^x+e^{-x}}{e^x-e^{-x}}=+\infty \quad \text{and} \quad \lim_{x\to+\infty} \mathrm{cth}_2 x=\lim_{x\to+\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}=1,$$

which, together with the Darboux property, implies that the function cth_2 is surjective. Then, because this function is continuous and strictly decreasing on the interval $(0,+\infty)$, it follows that she is injective. In conclusion, the function cth_2 is bijective and, thus, she is invertible.

***Otherwise*:** According to the first equality from (1,4′), if $x, y\in(0,+\infty)$, then the following equivalences hold:

$$\mathrm{cth}_2 x=\mathrm{cth}_2 y \Leftrightarrow \frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{e^y+e^{-y}}{e^y-e^{-y}} \Leftrightarrow \frac{e^{2x}+1}{e^{2x}-1}=\frac{e^{2y}+1}{e^{2y}-1}$$
$$\Leftrightarrow e^{2x+y}+e^{2y}-e^{2x}-1=e^{x+2y}+e^{2x}-e^{2y}-1 \Leftrightarrow e^{2x}=e^{2y} \Leftrightarrow x=y,$$

which shows that the function cth_2 is injective. The surjectivity of cth_2 we can deduce and such: consider a certain element $y\in(1,+\infty)$ and solve the equation:

(1) $\mathrm{cth}_2 x=y$,

i.e., according to the first equality from (1,4′):

$$\frac{e^x+e^{-x}}{e^x-e^{-x}}=y.$$

This equation becomes:

$$e^{2x}=\frac{y+1}{y-1}\in(1,+\infty),$$

and, thus,

(2) $x=\frac{1}{2}\cdot\ln\frac{y+1}{y-1}\in(0,+\infty)$.

Therefore, for every $y\in(1,+\infty)$, the equation (1) has a solution in the interval $(0,+\infty)$; so, the function cth_2 is surjective. From the last equality it follows that the inverse of this function is the function:

$$\mathrm{cth}_2^{-1} : (1,+\infty) \to (0,+\infty),$$

where, for every $x\in(1,+\infty)$,

$$\mathrm{cth}_2^{-1}(x)=\frac{1}{2}\cdot\ln\frac{x+1}{x-1}.$$

32) This statement follows from those discussed in paragraphs 30) and 31).

33) Indeed, because, according to the equalities (2.5) and (2.5′), for every $x\in\mathbf{R}$,

$$(\mathrm{sch}x)'=\left(\frac{1}{\mathrm{ch}x}\right)'=\left(\frac{2}{e^x+e^{-x}}\right)'=-2\cdot\frac{e^x-e^{-x}}{(e^x+e^{-x})^2},$$

it follows that, for every $x\in(-\infty,0]$, $(\mathrm{sch}x)'\geq0$ and, for every $x\in[0,+\infty)$, $(\mathrm{sch}x)'\leq0$; which shows that the function sch is strictly increasing on $(-\infty,0]$ and strictly decreasing on $[0,+\infty)$.

***Otherwise*:** According to the first equality from (2.5′), for every $x, y\in\mathbf{R}$,

$$\mathrm{sch}x-\mathrm{sch}y=\frac{2}{e^x+e^{-x}}-\frac{2}{e^y+e^{-y}}=\frac{2\cdot e^x}{e^{2x}+1}-\frac{2\cdot e^y}{e^{2y}+1}=2\cdot\frac{e^x\cdot(e^{2y}+1)-e^y\cdot(e^{2x}+1)}{(e^{2x}+1)\cdot(e^{2y}+1)}$$
$$=2\cdot\frac{e^{x+2y}+e^x-e^{2x+y}-e^y}{(e^{2x}+1)\cdot(e^{2y}+1)}=2\cdot\frac{e^{x+y}\cdot(e^y-e^x)+(e^x-e^y)}{(e^{2x}+1)\cdot(e^{2y}+1)}=2\cdot\frac{(e^x-e^y)\cdot(1-e^{x+y})}{(e^{2x}+1)\cdot(e^{2y}+1)}.$$

Now, if $x, y\in(-\infty,0]$, such that $x<y$, then $e^x<e^y$, $e^{x+y}<1$ and, thus, $\mathrm{sch}x<\mathrm{sch}y$, which shows that the function sch is strictly increasing on $(-\infty,0]$, and if $x, y\in[0,+\infty)$, such that $x<y$, then $e^x<e^y$, $e^{x+y}>1$ and, thus, $\mathrm{sch}x>\mathrm{sch}y$, which shows that the function sch is strictly decreasing on $[0,+\infty)$.

34) First we observe that the function:

$$\mathrm{sch}_1 : (-\infty,0] \to (0,1],$$

given from law: for every $x\in(-\infty,0]$,

$$\mathrm{sch}_1(x)=\mathrm{sch}x,$$

is continuous on the interval $(-\infty,0]$, because is a sum, respectively a fraction o two continuous functions, on this interval, according to the equalities (2.5′). So, the function sch_1 has the Darboux property on the interval $(-\infty,0]$. On the other hand,

$$\lim_{x\to-\infty} sch_1x=\lim_{x\to-\infty}\frac{2}{e^x+e^{-x}}=0 \qquad \text{and} \qquad \lim_{\substack{x\to0\\x<0}} sch_1x=\lim_{\substack{x\to0\\x<0}}\frac{2}{e^x+e^{-x}}=1,$$

which, together with the Darboux property, implies that the function sch_1 is surjective. Then, because this function is continuous and strictly increasing on the interval $(-\infty,0]$, it follows that she is injective. In conclusion, the function sch_1 is bijective and, thus, she is invertible.

***Otherwise*:** According to the first equality from (2.5′), for $x, y\in(-\infty,0]$, we have the equivalences:

$$sch_1x=sch_1y \Leftrightarrow \frac{2}{e^x+e^{-x}}=\frac{2}{e^y+e^{-y}} \Leftrightarrow \frac{e^x}{e^{2x}+1}=\frac{e^y}{e^{2y}+1} \Leftrightarrow e^{2x+y}+e^y=e^{x+2y}+e^x$$

$$\Leftrightarrow e^{2x+y}-e^{x+2y}+e^y-e^x=0 \Leftrightarrow e^{x+y}\cdot(e^x-e^y)-(e^x-e^y)=0 \Leftrightarrow (e^x-e^y)\cdot(e^{x+y}-1)=0 \Leftrightarrow x=y,$$

which shows that the function sch_1 is injective. The surjectivity of sch_1 we can deduce and such: consider a certain element $y\in(0,1]$ and solve the equation:

(1) $sch_1x=y$,

i.e., according to the first equality from (2.5′):

$$\frac{2}{e^x+e^{-x}}=y.$$

This equation becomes:

$$e^{2x}\cdot y-2\cdot e^x+y=0,$$

whence it follows that:

$$e^x=\frac{1-\sqrt{1-y^2}}{y}\in(0,1),$$

and, thus,

(2) $x=\ln\left(\frac{1-\sqrt{1-y^2}}{y}\right)\in(-\infty,0]$.

Therefore, for every $y\in(0,1]$, the equation (1) has a solution in the interval $(-\infty,0]$; so, the function sch_1 is surjective. From the equality (2) it follows that the inverse of this function is the function:

$$sch_1^{-1} : (0,1] \to (-\infty,0],$$

where, for every $x\in(0,1]$,

$$sch_1^{-1}(x)=\ln\left(\frac{1-\sqrt{1-y^2}}{y}\right).$$

35) First we observe that the function:

$$sch_2 : [0,+\infty) \to (0,1],$$

given from law: for every $x\in[0,+\infty)$,

$$sch_2(x)=schx,$$

is continuous pe intervalul $[0,+\infty)$, because is a sum, respectively a fraction o two continuous functions, on this interval, according to the equalities (2.5′). So, the function sch_2 has the Darboux property on the interval $[0,+\infty)$. On the other hand,

$$\lim_{\substack{x\to0\\x>0}} sch_2x=\lim_{\substack{x\to0\\x>0}}\frac{2}{e^x+e^{-x}}=1 \qquad \text{and} \qquad \lim_{x\to+\infty} sch_2x=\lim_{x\to+\infty}\frac{2}{e^x+e^{-x}}=0,$$

which, together with the Darboux property, implies that the function sch_2 is surjective. Then, because this function is continuous and strictly increasing on the interval $[0,+\infty)$, it follows that she is injective. In conclusion, the function sch_2 is bijective and, thus, she is invertible.

***Otherwise*:** According to the equalities (2.5′), for $x, y\in(-\infty,0]$, we have the equivalences:

$$sch_2x=sch_2y \Leftrightarrow \frac{2}{e^x+e^{-x}}=\frac{2}{e^y+e^{-y}} \Leftrightarrow \frac{e^x}{e^{2x}+1}=\frac{e^y}{e^{2y}+1} \Leftrightarrow e^{2x+y}+e^y=e^{x+2y}+e^x$$

$$\Leftrightarrow e^{2x+y}-e^{x+2y}+e^y-e^x=0 \Leftrightarrow e^{x+y}\cdot(e^x-e^y)-(e^x-e^y)=0 \Leftrightarrow (e^x-e^y)\cdot(e^{x+y}-1)=0 \Leftrightarrow x=y,$$

which shows that the function sch_2 is injective. The surjectivity of sch_2 we can deduce and such: consider a certain element $y\in(0,1]$ and solve the equation:

(1) $sch_2x=y$,

i.e., according to the first equality from (2.5′):

$$\frac{2}{e^x+e^{-x}}=y.$$

This equation becomes:

$$e^{2x}\cdot y-2\cdot e^x+y=0,$$

whence the it follows that:

$$e^x=\frac{1+\sqrt{1-y^2}}{y}\in[1,+\infty),$$

and, thus,

(2) $x=\ln\left(\frac{1+\sqrt{1-y^2}}{y}\right)\in[0,+\infty).$

Therefore, for every $y\in(0,1]$, the equation (1) has a solution in the interval $[0,+\infty)$; so, the function sch_2 is surjective. From the equality (2) it follows that the inverse of this function is the function:

$$sch_2^{-1}:(0,1]\to[0,+\infty),$$

where, for every $x\in(0,1]$,

$$sch_2^{-1}(x)=\ln\left(\frac{1+\sqrt{1-y^2}}{y}\right).$$

36) Indeed, because, according to the equalities (3.15), (2.6) and (2.6′), for every $x\in\mathbf{R}^*$, we have the following equalities:

$$(cshx)'=\left(\frac{1}{shx}\right)'=\left(\frac{2}{e^x-e^{-x}}\right)'=-2\cdot\frac{e^x+e^{-x}}{(e^x-e^{-x})^2}<0,$$

it follows that, for every $x\in\mathbf{R}^*$, the function csh is strictly decreasing both on $(-\infty,0)$, and the $(0,+\infty)$.

***Otherwise*:** According to the first equality from (2.6′), for every x, $y\in\mathbf{R}^*$, we have the equalities:

$$cshx-cshy=\frac{2}{e^x-e^{-x}}-\frac{2}{e^y-e^{-y}}=\frac{2\cdot e^x}{e^{2x}-1}-\frac{2\cdot e^y}{e^{2y}-1}=2\cdot\frac{e^x\cdot(e^{2y}-1)-e^y\cdot(e^{2x}-1)}{(e^{2x}-1)\cdot(e^{2y}-1)}$$

$$=2\cdot\frac{e^{x+2y}-e^x-e^{2x+y}+e^y}{(e^{2x}-1)\cdot(e^{2y}-1)}=2\cdot\frac{e^{x+y}\cdot(e^y-e^x)-(e^x-e^y)}{(e^{2x}-1)\cdot(e^{2y}-1)}=-2\cdot\frac{(e^x-e^y)\cdot(1+e^{x+y})}{(e^{2x}-1)\cdot(e^{2y}-1)}.$$

Now, if x, $y\in(-\infty,0)$, such that $x<y$, then $e^x<e^y$, $e^{2x}<1$, $e^{2y}<1$ and, thus, $cshx>cshy$, which shows that the function csh is strictly decreasing on $(-\infty,0)$, and if x, $y\in(0,+\infty)$, such that $x<y$, then $e^x<e^y$, $e^{2x}>1$, $e^{2y}>1$ and, thus, $cshx>cshy$, which shows that the function csh is strictly decreasing on $(0,+\infty)$.

37) First we observe that the function:

$$csh_1:(-\infty,0)\to(-\infty,0),$$

given from law: for every $x\in(-\infty,0)$,

$$csh_1(x)=cshx,$$

is continuous on the interval $(-\infty,0)$, because is a sum, respectively a fraction o two continuous functions, on this interval, according to the equalities (2.6′). So, the function csh_1 has the Darboux property on the interval $(-\infty,0)$. On the other hand,

$$\lim_{x\to-\infty} csh_1x=\lim_{x\to-\infty}\frac{2}{e^x-e^{-x}}=0 \quad\text{and}\quad \lim_{\substack{x\to0\\x<0}} csh_1x=\lim_{\substack{x\to0\\x<0}}\frac{2}{e^x-e^{-x}}=-\infty,$$

which, together with the Darboux property, implies that the function csh_1 is surjective. Then, because this function is continuous and strictly decreasing on the interval $(-\infty,0)$, it follows that she is injective. In conclusion, the function csh_1 is bijective and, thus, she is invertible.

***Otherwise*:** According to the first equality from (2.6′), for every x, $y\in(-\infty,0)$, the following equivalences hold:

$$\mathrm{csh}_1x=\mathrm{csh}_1y \Leftrightarrow \frac{2}{e^x-e^{-x}}=\frac{2}{e^y-e^{-y}} \Leftrightarrow \frac{e^x}{e^{2x}-1}=\frac{e^y}{e^{2y}-1} \Leftrightarrow e^{2x+y}-e^y=e^{x+2y}-e^x \Leftrightarrow e^{2x+y}-e^{x+2y}+e^x-e^y=0$$

$$\Leftrightarrow e^{x+y}\cdot(e^x-e^y)+(e^x-e^y)=0 \Leftrightarrow (e^x-e^y)\cdot(e^{x+y}+1)=0 \Leftrightarrow x=y,$$

which shows that the function csh_1 is injective. The surjectivity of csh_1 we can deduce and such: consider a certain element $y\in(-\infty,0)$ and solve the equation:

(1) $\mathrm{csh}_1x=y$,

i.e., according to the first equality from (2.6′):

$$\frac{2}{e^x-e^{-x}}=y.$$

This equation becomes:

$$e^{2x}\cdot y-2\cdot e^x-y=0,$$

whence it follows that:

$$e^x=\frac{1-\sqrt{1+y^2}}{y}\in(0,1),$$

and, thus,

(2) $x=\ln\left(\frac{1-\sqrt{1+y^2}}{y}\right)\in(-\infty,0).$

Therefore, for every $y\in(-\infty,0)$, the equation (1) has a solution in the interval $(-\infty,0)$; so, the function csh_1 is surjective. From the equality (2) it follows that the inverse of this function is the function:

$$\mathrm{csh}_1^{-1} : (-\infty,0) \to (-\infty,0),$$

where, for every $x\in(-\infty,0)$,

$$\mathrm{csh}_1^{-1}(x)=\ln\left(\frac{1-\sqrt{1+y^2}}{y}\right).$$

38) First we observe that the function:

$$\mathrm{csh}_2 : (0,+\infty) \to (0,+\infty),$$

given from law: for every $x\in(0,+\infty)$,

$$\mathrm{csh}_2(x)=\mathrm{csh}x,$$

is continuous on the interval $(0,+\infty)$, because is a sum, respectively a fraction o two continuous functions, on this interval, according to the equalities (2.6′). So, the function csh_2 has the Darboux property on the interval $(0,+\infty)$. On the other hand,

$$\lim_{x\to+\infty}\mathrm{csh}_2x=\lim_{x\to+\infty}\frac{2}{e^x-e^{-x}}=0 \quad \text{and} \quad \lim_{\substack{x\to0\\x>0}}\mathrm{csh}_2x=\lim_{\substack{x\to0\\x>0}}\frac{2}{e^x-e^{-x}}=+\infty,$$

which, together with the Darboux property, implies that the function csh_2 is surjective. Then, because this function is continuous and strictly decreasing on the interval $(0,+\infty)$, it follows that she is injective. In conclusion, the function csh_2 is bijective and, thus, she is invertible.

***Otherwise*:** According to the first equality from (2.6′), dacă x, $y\in(0,+\infty)$, then the following equivalences hold:

$$\mathrm{csh}_2x=\mathrm{csh}_2y \Leftrightarrow \frac{2}{e^x-e^{-x}}=\frac{2}{e^y-e^{-y}} \Leftrightarrow \frac{e^x}{e^{2x}-1}=\frac{e^y}{e^{2y}-1} \Leftrightarrow e^{2x+y}-e^y=e^{x+2y}-e^x \Leftrightarrow e^{2x+y}-e^{x+2y}+e^x-e^y=0$$

$$\Leftrightarrow e^{x+y}\cdot(e^x-e^y)+(e^x-e^y)=0 \Leftrightarrow (e^x-e^y)\cdot(e^{x+y}+1)=0 \Leftrightarrow x=y,$$

which shows that the function csh_2 is injective. The surjectivity of csh_2 we can deduce and such: consider a certain element $y\in(0,+\infty)$ and solve the equation:

(1) $\mathrm{csh}_2x=y$,

i.e, according to the first equality from (2.6′):

$$\frac{2}{e^x-e^{-x}}=y.$$

This equation becomes:

$e^{2x}\cdot y-2\cdot e^{x}-y=0$,

whence it follows that:

$$e^{x}=\frac{1+\sqrt{1+y^2}}{y}\in(1,+\infty),$$

and, thus,

(2) $$x=\ln\left(\frac{1+\sqrt{1+y^2}}{y}\right)\in(0,+\infty).$$

Therefore, for every $y\in(0,+\infty)$, the equation (1) has a solution in the interval $(0,+\infty)$; so, the function csh_2 is surjective. From the equality (2) it follows that the inverse of this function is the function:

$$csh_2^{-1}:(0,+\infty)\to(0,+\infty),$$

where, for every $x\in(0,+\infty)$,

$$csh_2^{-1}(x)=\ln\left(\frac{1+\sqrt{1+y^2}}{y}\right). \square$$

4. Conclusions

As you can see, in this paper we presented definitions of hyperbolic functions, immediate properties and 38 other properties, divided into four groups. The aim was to form the reader's attention and interest in these issues first global image of these functions, that would help in addressing other issues more complex. Precisely why the the demonstrations are presented in full, in detail, so that it can be used in the classroom.

References

[1] Abramowitz, M. and Stegun, I. A. (Eds.) (1973), *Hyperbolic Functions* §4.5 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, p. 83-86.

[2] Anderson, J. W., (1999), *Trigonometry in the Hyperbolic Plane* §5.7 in Hyperbolic Geometry, New York: Springer-Verlag, p. 146-151.

[3] Beyer, W. H., (1987), *Hyperbolic Function* CRC Standard Mathematical Tables, 28th ed. Boca Raton, FL: CRC Press, p. 168-186 and 219.

[4] Coxeter, H. S. M., Greitzer, S. L., (1967), *Geometry Revisited*, Washington, DC: Math. Assoc. Amer., p. 126-131.

[5] Harris, J. W., Stocker, H., (1998), *Hyperbolic Functions*, Handbook of Mathematics and Computational Science, New York, Springer-Verlag, p. 245-262.

[6] Jeffrey, A., (2000), *Hyperbolic Identities*, §3.5 in Handbook of Mathematical Formulas and Integrals, 2nd ed. Orlando, FL: Academic Press, p. 117-122.

[7] Norden, H. P., (1953), *Introducere elementară în geometria lui Lobacevski (Elementary introduction in geometry of Lobacevski)*, Bucureşti, Editura Tehnică de Stat, Bucureşti, 1953.

[8] Yates, R. C., (1953), *Hyperbolic Functions*, A Handbook on Curves and Their Properties. Ann Arbor, MI, J. W. Edwards, p. 113-118.

[9] Zwillinger, D. (Ed.), (1995), *Hyperbolic Functions*, §6.7 in CRC Standard Mathematical Tables and Formulae. Boca Raton, FL, CRC Press, p. 476-481.

[10] https://www.coastal.edu/mathcenter/HelpPages/Handouts/invhyp.PDF

[11] http://home.scarlet.be/math/hyp.htm:Hyperbolic functions and inverses

[12] http://wwwf.imperial.ac.uk/metric/metric_public/functions_and_graphs/hyperbolic_functions/inver ses.html

[13] http://www.mathcentre.ac.uk/resources/workbooks/mathcentre/hyperbolicfunctions.pdf

[14] http://scimath.unl.edu/MIM/files/MATExamFiles/Schutz_MATpaper_FINAL.pdf

[15] http://tartarus.org/gareth/maths/stuff/hyperbolic_functions.pdf

[16] https://www.whitman.edu/mathematics/calculus_online/section04.12.html

Author

Teodor Dumitru Vălcan, „*Babeș-Bolyai*” University, Cluj-Napoca (Romania).

E-mail: tdvalcan@yahoo.ca.